

Symplectic Quantum Mechanics

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Abstract

We propose $Sp(8, \mathbb{R})$ and $SO(9, \mathbb{R})$ as dynamical groups for closed quantum systems. Restricting here to $Sp(8, \mathbb{R})$, the quantum theory is constructed and investigated. The functional Mellin transform plays a prominent role in defining the quantum theory as it provides a bridge between the quantum algebra of observables and the algebra of operators on Hilbert spaces furnishing unitary representations that are induced from a distinguished parabolic subgroup of $Sp(8, \mathbb{R})$. As well, the parabolic subgroup furnishes a fiber bundle construction that models what can be described as a matrix quantum gauge theory. The formulation is strictly quantum mechanics: no *a priori* space-time is assumed and the only geometrical input comes from the group manifold. But, what appears on the surface to be a fairly simple model, turns out to have a capacious structure suggesting some surprising physical interpretations.

1 Introduction

1.1 Motivation

The program of quantization in quantum mechanics (QM) is usually approached from the bottom up. That is, physical considerations identify a classical phase space — whose points represent classical states — and suitable functions on that phase space. Then one attempts to promote: (i) the classical states to a suitable Hilbert space, and (ii) the phase space functions to suitable operators on that Hilbert space.

The obvious alternative is a top-down approach. Here the goal is to construct, either algebraically or functionally, a C^* -algebra and an associated Hilbert space. Of course, the key is to somehow find the correct formulation ‘up stairs’, since generally it doesn’t correspond one-to-one with the ‘down stairs’ classical theory.

Our tack in this paper is a top-down approach that mingles both functional and algebraic constructs. The idea is to model the C^* -algebra \mathfrak{A} that characterizes a quantum system by a C^* -algebra of integrable functionals $\mathbf{F}(G^{\mathbb{C}}) \ni F : G^{\mathbb{C}} \rightarrow L_B(\mathcal{H})$ where: $G^{\mathbb{C}}$ is the complexification of a generically non-compact topological group isomorphic to the group of units of \mathfrak{A} , the Hilbert space \mathcal{H} furnishes a direct sum of

all relevant unitary irreducible representations of $G^{\mathbb{C}}$, and $L_B(\mathcal{H})$ is the C^* -algebra of linear bounded operators on \mathcal{H} . The two C^* -algebras $\mathbf{F}(G^{\mathbb{C}})$ and $L_B(\mathcal{H})$ are dual to each other via the functional Mellin transform (to be explained later), and $F \in \mathbf{F}(G^{\mathbb{C}})$ is said to be integrable if the associated functional integral is well-defined (also to be explained later).

This approach allows to view the main task of quantization as a choice of some topological group G . The topological group simultaneously generates: (i) the Hilbert space of states, (ii) the C^* -algebra of integrable functionals, and (iii) the evolution dynamics.[1]

It is important to emphasize that G is typically non-compact. Insofar as a quantum system must be quantified through observation/measurement, G must therefore be inferred from homomorphic *locally compact* topological groups. The idea is that observation of a quantum system leads to a continuous homomorphism $\lambda : G \rightarrow G_\lambda$ with G_λ a locally compact topological group. Consequently, G is indirectly identified with an entire family $G_\Lambda := \{G_\lambda, \lambda \in \Lambda\}$ where the set Λ characterizes all possible ‘localizations’ $\lambda : G \rightarrow G_\lambda$ of a given system.¹ These ‘localizations’ embody the Born rule by reducing pertinent functional integrals to bona fide integrals.

Given its significant responsibility, one would expect a judicious choice of G_λ would lead to interesting and relevant physics if the functional approach is indeed valid. So the obvious next step in the program initiated in [1] is to determine or otherwise guess G_λ and then study the resulting QM.

We first give a brief motivation regarding our guess for G_λ . Begin with the simple idea that dynamical interactions can be modeled by correlations among a set of ‘constituents’². Consider a physical quantum system composed of N constituents. At a purely formal level, an analysis of the correlations representing interactions among N constituents leads to the identification of $U(N)$ as an organizing group. Under various circumstances, certain correlations will dominate others, and this will induce sub-organizations in $U(N)$ referred to as subduction. Evidently this subduction will continue to be driven by dynamics until some (quasi)equilibrium organizing group, which we will call the dynamical group, is achieved that describes the system correlations. A few moments reflection on the subduction of $U(N)$ into its relevant subgroup chains (along with some liberal hand-waving) leads to just two parent groups that seem to be viable candidates for a quantum dynamical group — $Sp(2n, \mathbb{R})$ and $SO(2n + 1, \mathbb{R})$ — depending on odd or even permutation symmetry. Presumably, parameter n represents surviving constituents (degrees of freedom) that encode the correlations between the original N constituents.

Although it is likely that $SO(2n, \mathbb{R})$, $SO(2n + 1, \mathbb{R})$ and $Sp(2n, \mathbb{R})$ are viable dynamical groups for some interesting physical systems (especially for $n < 4$), it is our contention that $Sp(8, \mathbb{R})$ and $SO(9, \mathbb{R})$ are the most relevant for the majority of

¹We are skipping a good deal of detail and explanation here that can be found in [1] and [2].

²By ‘constituents’ we mean excitations of a quantum system that are (quasi)stationary in time during an evolution.

physical observation at typical terrestrial energy densities. Our hand-waving motivation aside, we therefore *postulate* that $Sp(8, \mathbb{R})$ and $SO(9, \mathbb{R})$ are dynamical groups that govern (some) quantum systems.³

To keep the exposition manageable, we will restrict attention to $Sp(8, \mathbb{R})$ in this paper and investigate only symplectic quantum mechanics (SQM). Whether or not $Sp(8, \mathbb{R})$ describes realistic quantum systems and reduces to realistic classical systems is of course paramount. We will present evidence and argue that it does, but obviously the issue cannot be settled in a single paper.

1.2 The symplectic case

Symplectic symmetry is no stranger to classical mechanics, and the suspicion that symplectic groups have importance as *dynamical* groups for quantum mechanics⁴ has been around for a long time — for obvious reasons based on the correspondence principle and the close relationship between the symplectic and Heisenberg groups (see e.g. [3]). Many of these works rely on the discrete series representation of $Sp(2n, \mathbb{R})$. This has advantages and disadvantages: It allows one to use familiar methods involving raising and lowering operators on discrete number-type states. On the other hand, in some applications the physical interpretation of the discrete states is not always manifest. In particular, it is not clear how to interpret the quantum numbers of the discrete states for $Sp(8, \mathbb{R})$. Nevertheless, the idea of symplectic dynamical symmetry continues to attract attention and produce reasonable successes; most notably perhaps in nuclear physics (see [4] and the references therein).⁵

Having settled on $Sp(8, \mathbb{R})$ as a dynamical group, the first task is to define the quantum theory. For this we utilize and assume familiarity with [1]. Applying the quantization scheme proposed in [1] invokes three key notions: (i) $Sp(8, \mathbb{R})$ contains a distinguished parabolic subgroup that determines relevant induced unitary representations which are then used to construct the quantum Hilbert space. (ii) A C^* -algebra containing quantum observables is constructed via functional Mellin transforms. Together with the Hilbert space, this provides the kinematic backdrop of the quantum theory, and it allows concrete functional integral realizations of interesting quantum operators. (iii) Inner automorphisms of the symplectic group induce inner automorphisms of the C^* -algebra that yield system dynamics.

Assuming the quantum theory is well-defined by this construction, we move on to

³Because of the close connection between $Sp(8, \mathbb{R})$ and $SO(9, \mathbb{R})$, it is tempting to bring them together under $OSp(9, 8)$ to incorporate supersymmetry. However, our hunch is that $OSp(9, 8)$ is rather like an unstable point in a hypothetical space of dynamical groups, and that observable systems only pass through $OSp(9, 8)$ under certain evolutions that change the orthogonal/symplectic probabilistic character of the system correlations.

⁴The compact form of the symplectic group shows up in string theory too, but we are restricting attention to dynamical groups in the context of simple QM here.

⁵ $Sp(2n, \mathbb{R})$ has been investigated in the context of quantum optics [5], [6]; but as a symmetry of canonical commutation relations not as a dynamical group.

interpret and explore some physical implications.

An important attribute of $Sp(8, \mathbb{R})$ is it possesses a parabolic subgroup P with $\dim_{\mathbb{R}}(P) = 26$ that contains the maximally compact subgroup $U(4) \subset P$. Since $Sp(8, \mathbb{R})$ is supposed to be dynamical and the group elements that are not contained in P mutually commute, we propose that P can be interpreted as a local gauge group responsible for internal forces associated with $U(4)$ and external forces associated with $P/U(4)$.

At the Lie algebra level, $\mathfrak{Sp}(8)$ comprises 36 generators; ten of which mutually commute. Of the remaining 26 that generate the parabolic subgroup P , sixteen span the algebra $\mathfrak{U}(4)$.⁶ The remaining ten generators of P , in combination with the only two involutive inner automorphisms of the algebra, ultimately give rise to ‘expected geometry’. More precisely, the ground-state expectation values of the associated operators characterize the geometry of a complex manifold parametrized by the spectrum of the commuting observables associated with the homogenous space $Z = Sp(8, \mathbb{R})/P$ (under suitable conditions).

According to the manifold structure of $Sp(8, \mathbb{R})$, there are five local domains where the maximal torus has signatures $(0, 4)$, $(1, 3)$, $(2, 2)$, $(3, 1)$ and $(4, 0)$.^[7] The odd signatures give rise to what may be interpreted as ‘expected space-time’ for the configuration space of a certain cotangent bundle. (The physical meaning of the even signatures is unclear.) Since the observables associated with Z and P are dynamical, the cotangent bundle is likewise. Consequently, space-time is interpreted as the ground-state expectation of certain operators in this model, and it is dynamical.

There is nothing special about this notion of expected geometry: a similar statement could be made regarding the Heisenberg group in standard non-relativistic quantum mechanics. That is, VEVs of the CCR could be interpreted as an ‘expected cotangent bundle’ with suitable choice of representation. The difference is that the expected geometry of $Sp(8, \mathbb{R})$ leads to a 10-d configuration space with a $U(4)$ internal symmetry. The ten dimensions aside, this difference is significant because it mixes the configuration parameters and $U(4)$ charges. We will see later that this means the associated operators create/annihilate $U(4)$ charges at different points in Z . This can be interpreted as particle creation/annihilation which of course does not happen in standard non-relativistic QM. It also means that matter (understood as the presence of $U(4)$ charge) and geometry are inseparable at the quantum algebra level.

The parabolic subgroup plays a second important role: it is the basis of induced representations which find a natural description in a fiber bundle framework. The induced representations ultimately form the quantum Hilbert space of states and the associated fiber bundle geometry gives a coherent state⁷ model of state-vectors and

⁶The standard notation for these Lie algebras would be $\mathfrak{sp}(8)$ and $\mathfrak{u}(8)$. We choose to maintain the capitalization of the associated Lie group to more clearly differentiate between the algebra and its elements for generic groups. For example, for a group we write $g \in G$ and for its algebra $\mathfrak{g} \in \mathfrak{G}$.

⁷The term coherent state is a bit imprecise, but we will conform to standard usage. Strictly, for us a coherent state is a model of a Hilbert *state-vector* (as opposed to a projective-Hilbert *state*) on a (sub)manifold of some Lie group.

operators. It is in the coherent state formulation where the physical interpretation of the theory begins to emerge: one can interpret/identify ground states, matter particles, gauge particles, and their associated fields, and exhibit explicit realizations of relevant operators.⁸ With these objects, meaningful transition amplitudes can be constructed and interpreted.

Again, there is nothing special about inducing representations: the same approach is used for the non-compact Poincaré group. In the case of Poincaré, the little group and the mutually commuting momentum generators yield spin/helicity and momenta labels for state-vectors along with a particle creation/annihilation interpretation; and they induce an ‘expected fiber bundle’ structure with a momentum base space and Lorentz structure group.⁹ The crucial difference brought by $Sp(8, \mathbb{R})$ is the mingling of ‘internal’ and ‘external’ symmetries resulting in a dynamical cotangent bundle structure.

Besides providing physical interpretation via a distinguished parabolic structure, the symplectic group is supposed to govern system dynamics through inner automorphisms. It turns out that the CS model of such dynamics in the Heisenberg picture is matrix quantum mechanics — which is known to have a deceptively intricate structure. For example, it is known that in the adjoint representation the dynamics of the ten commuting operators of the Lie algebra approach a membrane theory for large systems. The remaining operators that generate the parabolic subgroup represent gauge degrees of freedom, and the adjoint representation provides a matrix gauge theory interpretation. The gauge theory is not exactly a Yang-Mills gauge theory because the parabolic subgroup is non-compact: nevertheless, it possesses unitary induced representations. In fact, the group is contained in the units of the C^* -algebra, so it should be possible to formulate the dynamics as $U(4)$ Yang-Mills on a non-commutative phase space. But then one must interpret the physical meaning of the cotangent-operators.¹⁰

Because the dynamics is governed by inner automorphisms, evolution transforms the parabolic subgroup P according to the adjoint action. In consequence, for non-trivial dynamics determined by some $h(t) \in Sp(8, \mathbb{R})$, a new more relevant parabolic subgroup $\tilde{P} = Ad(h(t))P$ can emerge. Hence, a new non-trivial ground state may be associated with the final state of an evolution process by a suitable choice of representation.¹¹ The catch is that one must somehow relate the original ground state

⁸Although indirectly related, the coherent state model of fields and operators are not the same fields and operators of a QFT.

⁹For Poincaré, one can then construct an ‘expected cotangent bundle’ by attaching a *static* space-time to the ‘expected fiber bundle’ base space generated by the momentum operators. Recall that the boost generators contain all the dynamics induced by the Hamiltonian operator: the 4-momentum and angular-momentum generators are inert. Accordingly, the cotangent bundle is static, and it would seem the dynamics associated with boost can be naturally interpreted as inertia.

¹⁰Unfortunately, we have only incomplete results that are not ready to report here.

¹¹The notion of a CS model with a non-trivial vacuum is similar in spirit to an effective theory of quasi-particles. However there is an important difference: all vacua (along with their associated

representation to the non-trivial ground state representation in order to relate and interpret the physical properties of the corresponding coherent states. Nevertheless, the underlying framework is a quantum theory. It has an adjustable ground state, and it applies equally to all resolution scales; micro-, meso-, and macroscopic systems.

Having proposed interpretations of various objects in the quantum theory, one would like to compare with corresponding classical objects. The associated classical dynamics is defined via the correspondence principle, and the classical Poisson bracket turns out to be the large-system (i.e. many-particle) limit of the Lie algebra-induced bracket on the C^* -algebra. The result is classical Hamiltonian mechanics on a cotangent bundle with a 10-d configuration space. Six of the dimensions are interpreted as directed-area degrees of freedom, and there is no incentive to compactify them. Indeed, they appear to have important physical significance: they define a volume element in a 4-d space, and they may represent vortex-like degrees of freedom.

One final point; $Sp(8, \mathbb{R})$ has discrete series and holomorphic discrete series representations. So at the end of the day, one can throw out all the physical motivation and interpretation supplied by the parabolic decomposition and its concomitant coherent state model and simply refer everything to the discrete series representation(s). Presumably, this would constitute a rigorous formulation.

2 Quantization

2.1 Representations

It is appropriate to begin with a review of the structure of the symplectic algebra pointing out some of its implications and then to construct unitary representations (ureps) that are particularly amenable to physical interpretation.

2.1.1 Symplectic Lie algebra

$Sp(8, \mathbb{R})$ is a rank-4 reductive Lie group of $\dim_{\mathbb{R}}(Sp(8, \mathbb{R})) = 36$. The first order of business is to examine the structure of the adjoint representation to learn what type of dynamics it encodes.

Consider the triangular decomposition of its Lie algebra;

$$\mathfrak{Sp}(8) =: \mathfrak{G} = \mathfrak{G}_- \oplus \mathfrak{G}_0 \oplus \mathfrak{G}_+ \quad (2.1)$$

where

$$\begin{aligned} [\mathfrak{G}_0, \mathfrak{G}_0] &= 0 \\ [\mathfrak{G}_+, \mathfrak{G}_-] &\subseteq \mathfrak{G}_0 \\ [\mathfrak{G}_{\pm}, \mathfrak{G}_0 \oplus \mathfrak{G}_{\pm}] &\subseteq \mathfrak{G}_{\pm} . \end{aligned} \quad (2.2)$$

CS) are related via evolution through $Sp(8, \mathbb{R})$.

This decomposition is physically relevant, because in the adjoint representation it defines charges associated with the symmetry that can be used to characterize states if the associated quantum system respects the symmetry. The subalgebra \mathfrak{G}_0 contains neutral states and \mathfrak{G}_\pm contains charged states associated with ‘particles’ and ‘anti-particles’. In our case, there are four neutral states and 32 charged states characterized by various combinations of four types of charge.

To render these brackets more explicit, let S_F denote a Fock space of *bosonic* excitations above some vacuum. Define creation and annihilation operators acting on this space by

$$c_\alpha c_\beta^\dagger - c_\beta^\dagger c_\alpha = \delta_{\alpha\beta} \quad ; \quad c_\alpha^\dagger c_\beta^\dagger - c_\beta^\dagger c_\alpha^\dagger = 0 \quad ; \quad c_\alpha c_\beta - c_\beta c_\alpha = 0 \quad (2.3)$$

where $\alpha, \beta \in \{\pm 1, \dots, \pm 4\}$ and \dagger indicates the conjugate operator. With these operators, a basis of $\mathfrak{Sp}(8)$ can be realized as[10]

$$c_{\alpha,\beta} := \frac{1}{\sqrt{1 + \delta_{\alpha,-\beta}}} \left(c_\alpha^\dagger c_\beta - (-1)^{\alpha-\beta} c_{-\beta}^\dagger c_{-\alpha} \right) \quad (2.4)$$

and rearranged as

$$\begin{aligned} \mathfrak{h}_i &:= c_{i,i} = n_i - n_{-i} \\ \mathfrak{e}_i &:= c_{i,-i} = \sqrt{2} c_i^\dagger c_{-i} \\ \mathfrak{e}_{-i} &:= c_{-i,i} = \mathfrak{e}_i^\dagger \\ \mathfrak{e}_{i,j} &:= c_{i,(i-j)} \\ \mathfrak{e}_{j,i}^\dagger &= \mathfrak{e}_{i,j} = (-1)^{i+j} c_{(j-i),-i} \end{aligned} \quad (2.5)$$

where $n_i := c_i^\dagger c_i$, the indices $i, j \in \{1, \dots, 4\}$, and $i \neq j$. This arrangement characterizes the Borel subgroup and its induced coset space with associated subalgebras $\mathfrak{G}_0 \simeq \text{span}_{\mathbb{R}}\{\mathfrak{h}_i\}$, $\mathfrak{G}_+ \simeq \text{span}_{\mathbb{R}}\{\mathfrak{e}_i, \mathfrak{e}_{ij}\}$, and $\mathfrak{G}_- \simeq \mathfrak{G}_+^\dagger$.

The Borel decomposition can be used to build up an associated Fock space of states giving rise to the (infinite-dimensional) irreducible discrete series representations of $Sp(8, \mathbb{R})$. The states clearly represent excitations due to the action of $\mathfrak{Sp}(8)$, but the physical interpretation of the excitations and their associated charges is unclear.

Instead, we will choose a parabolic decomposition motivated by the fact that $U(4)$ is the maximal compact subgroup and the observation that there are ten mutually commuting generators contained in $\mathfrak{G}_- \oplus \mathfrak{G}_+$.

Consider

$$\begin{aligned} \{\mathfrak{i}_{ij}\} &:= \{\mathfrak{h}_i, \mathfrak{e}_{i,-j}, \mathfrak{e}_{i,-j}^\dagger\}, \quad 1 \leq i < j \leq 4 \\ \{\mathfrak{e}_{ij}, \mathfrak{e}_{ij}^\dagger\} &:= \left\{ (\mathfrak{e}_i, \mathfrak{e}_{i,+j}), (\mathfrak{e}_i^\dagger, \mathfrak{e}_{i,+j}^\dagger) \right\}, \quad 1 \leq i < j \leq 4. \end{aligned} \quad (2.6)$$

The first set generates $U(4)$, and the set of generators $\{\mathbf{e}_{ij}\}$ (resp. $\{\mathbf{e}_{ij}^\dagger\}$) mutually commute. They satisfy the commutation relations

$$\begin{aligned} [\mathbf{e}_{ij}, \mathbf{e}_{kl}^\dagger] &= \delta_{ik}\mathbf{i}_{lj} + \delta_{il}\mathbf{i}_{kj} + \delta_{jk}\mathbf{i}_{li} + \delta_{jl}\mathbf{i}_{ki} \\ [\mathbf{i}_{ij}, \mathbf{i}_{kl}] &= \delta_{jk}\mathbf{i}_{il} - \delta_{il}\mathbf{i}_{kj} \\ [\mathbf{i}_{ij}, \mathbf{e}_{kl}] &= \delta_{jk}\mathbf{e}_{il} + \delta_{jl}\mathbf{e}_{ik} \\ [\mathbf{i}_{ij}, \mathbf{e}_{kl}^\dagger] &= -\delta_{ik}\mathbf{e}_{jl}^\dagger - \delta_{il}\mathbf{e}_{jk}^\dagger \\ [\mathbf{e}_{ij}, \mathbf{e}_{kl}] &= [\mathbf{e}_{ij}^\dagger, \mathbf{e}_{kl}^\dagger] = 0. \end{aligned} \quad (2.7)$$

Let $\varrho' : \mathfrak{Sp}(8) \rightarrow L(\mathcal{V})$ be a representation with \mathcal{V} a \mathfrak{G} -module. The physically relevant triangular decomposition of the algebra induces a decomposition of \mathcal{V} by

$$\mathcal{V} = \bigoplus_w \mathcal{V}_{(w)}, \quad \mathcal{V}_{(w)} := \{\mathbf{v} \in \mathcal{V} : \varrho'(\mathbf{h}_i)\mathbf{v} = w_i\mathbf{v}\}, \quad i \in \{1, \dots, 4\} \quad (2.8)$$

where $\mathbf{h}_i \in \mathfrak{G}_0$ and $w = \{w_1, \dots, w_4\}$ is a weight in the basis of fundamental weights composed of complex eigenvalues $w_i \in \mathbb{C}$. It is well-known that a particular \mathcal{V} can be generated by acting with raising operators $\mathfrak{g}_+ \in \mathfrak{G}_+$ on a dominant-integral lowest-weight vector \mathbf{v}_{w_-} . Call this vector space \mathcal{V}_{w_-} .

Now, there is a distinguished subalgebra of \mathfrak{G} : its maximal compact subalgebra $\mathfrak{u}(4)$. Let $\mathcal{V}_{(\mu)} \subset \mathcal{V}_{w_-}$ denote the submodule generated by $\mathfrak{u}(4)$ acting on the dominant-integral lowest-weight vector \mathbf{v}_{w_-} . The submodule $\mathcal{V}_{(\mu)}$ then furnishes an irreducible representation (irrep) of $U(4)$ where $\mu = [\mu_1, \dots, \mu_4]$ is a partition based on w_- that labels the representation. Since w_- is a lowest weight, $\mathcal{V}_{(\mu)}$ is an invariant subspace with respect to the subalgebra $\mathfrak{P} := \mathfrak{G}_- \cup \mathfrak{u}(4)$, i.e. $\bar{\varrho}'(\mathfrak{P})\mathcal{V}_{(\mu)} \subseteq \mathcal{V}_{(\mu)}$ where $\bar{\varrho}'$ is a restricted representation of ϱ' . From this, one constructs reps of \mathfrak{P} based on lowest-weight irreps of $U(4)$ and labeled by partitions $[\mu_1, \dots, \mu_4]$.

Remark 2.1 *To get a glimpse of the physical content of this decomposition, combine the set $\{\mathbf{e}_{ij}, \mathbf{e}_{ij}^\dagger\}$ according to $\mathbf{q}_i := 1/2(\mathbf{e}_i + \mathbf{e}_i^\dagger)$ and $\mathbf{q}_{i,j} := 1/2(\mathbf{e}_{i,+j} + \mathbf{e}_{i,+j}^\dagger)$, and $\mathbf{p}_i := 1/2(\mathbf{e}_i - \mathbf{e}_i^\dagger)$ and $\mathbf{p}_{i,j} := 1/2(\mathbf{e}_{i,+j} - \mathbf{e}_{i,+j}^\dagger)$. Consider a discrete series representation with a degenerate partition $[\mu, \mu, \mu, \mu]$, and identify its lowest weight \mathbf{v}_{w_-} with the vacuum of some quantum system. It is straightforward to see that*

$$\langle \varrho'([\mathbf{q}_i, \mathbf{p}_j]) \rangle_{\mathbf{v}_{w_-}} \propto \delta_{ij} \quad ; \quad \langle \varrho'(\mathbf{p}_{i,j}) \rangle_{\mathbf{v}_{w_-}} \propto \epsilon_{ij}. \quad (2.9)$$

Moreover, it is easy to imagine the symmetry reduction $\mathfrak{u}(4) \rightarrow \mathfrak{u}(3) \times \mathfrak{u}(1)$.

We will return to this aspect in more detail later and argue that the parabolic decomposition can lead to a phase space with 4-d space-time and internal $U(4)$ gauge symmetry.

At the algebra level, the parabolic decomposition is described by

$$\mathfrak{Sp}(8) = \mathfrak{z}_- \oplus \mathfrak{u}(4) \oplus \mathfrak{z}_+ \quad (2.10)$$

where $\mathfrak{Z}_+ = \text{span}\{\mathbf{e}_{ij}\}$, $\mathfrak{Z}_- = \text{span}\{\mathbf{e}_{ij}^\dagger\}$ and $\mathfrak{U}(4) = \text{span}\{\mathbf{i}_{ij}\}$. Choosing the lowest weight as opposed to the highest weight vector to generate $\mathcal{V}_{(\mu)}$ seems arbitrary from a physical standpoint. Therefore, at least in this paper, we will assume that the system enjoys the symmetry $\mathfrak{Z}_+ \rightleftharpoons \mathfrak{Z}_-$.

The parabolic decomposition leads to a canonically associated coset space $Z := Sp(8, \mathbb{R})/P$ with $\dim_{\mathbb{R}}(Z) = 10$ generated by the subalgebra

$$\mathfrak{Z}_+ := \frac{\mathfrak{Sp}(8)}{\mathfrak{P}} := \frac{\mathfrak{Sp}(8)}{\mathfrak{Z}_- \oplus \mathfrak{U}(4)} . \quad (2.11)$$

Since the elements of \mathfrak{Z}_+ mutually commute, interpret Z as the system configuration space and the elements of \mathfrak{P} as the generators of ‘external’ and ‘internal’ dynamics. Accordingly, we propose to associate dynamical variables parametrizing ‘external’ interactions with the coset space $Sp(8, \mathbb{R})/U(4)$. This is a $\dim_{\mathbb{R}}(Sp(8, \mathbb{R})/U(4)) = 20$ manifold that can be interpreted as the system phase space.

The coset space Z furnishes both a convenient physical interpretation and the means to construct induced ureps.

2.1.2 Induced ureps

So far we have only considered the algebra $\mathfrak{Sp}(8)$. But what we need to find is *unitary* representations (ureps) of the group $Sp(8, \mathbb{R})$. This is a non-compact group, and we can’t simply exponentiate a representation of its algebra because relevant representations are generally infinite-dimensional in this case. The method we will use to construct ureps relies on Mackey’s theory of induced representations which has been thoroughly developed [17]–[25]. We give only an outline of the steps for a quantum system invariant under the involution $\mathfrak{Z}_- \rightleftharpoons \mathfrak{Z}_+$:

step 1: Find the *basic* dominant-integral lowest-weight modules of $\mathfrak{Sp}(8)$. There are four: $\{\mathcal{V}_1^{(0)}, \mathcal{V}_8^{(1)}, \mathcal{V}_{27}^{(2)}, \mathcal{V}_{48}^{(3)}, \mathcal{V}_{42}^{(4)}\}$ where the subscript denotes the dimension of the module. The trivial representation $\mathcal{V}_1^{(0)}$ has been included in this list, because it will represent the quantum vacuum. The defining module is $\mathcal{V}_8^{(1)}$, and the adjoint module is $\mathcal{V}_1^{(0)} \oplus \mathcal{V}_8^{(1)} \oplus \mathcal{V}_{27}^{(2)}$. Whether there are other relevant reps based on $\mathcal{V}_{48}^{(3)}$ and $\mathcal{V}_{42}^{(4)}$ is unclear, but there is no reason not to expect them.

step 2: For each relevant rep, identify the dominant-integral lowest-weight vector and generate the \mathfrak{P} invariant subspace $\mathcal{V}_{(\mu)} \subset \mathcal{V}_{w_-}$ for all relevant unitary irreps of $U(4)$ by acting on the dominant-integral lowest-weight vector \mathbf{v}_{w_-} . $U(4)$ being compact, its unitary irreps have finite dimension, and, since they are dominant-integral, the various $\mathcal{V}_{(\mu)}$ possess a positive definite hermitian inner product.

step 3: As an intermediate step, consider the principal bundle associated with the ‘internal’ symmetry; $(Sp(8, \mathbb{R}), Sp(8, \mathbb{R})/U(4), \check{p}r, U(4))$ and its associated vector bundle $(\mathcal{I}, Sp(8, \mathbb{R})/U(4), pr, \mathcal{V}_{(\mu)}, U(4))$. Note that cross sections of each bundle are canonically related and represent $U(4)$ degrees of freedom.

step 4: Recall that the action of \mathfrak{P} leaves $\mathcal{V}_{(\mu)}$ invariant. So maximum efficiency (associated with the impending induced representation) obtains through the factorization $Sp(8, \mathbb{R})/P$ which utilizes the finite-dimensional $\mathcal{V}_{(\mu)}$. Accordingly, trivially extend¹² the relevant unirreps of $U(4)$ to P . Since the ten elements in the factor algebra \mathfrak{Z}_+ mutually commute, we can anticipate that they yield a basis for compatible quantum observables.

step 5: Construct the principal coset bundle $(\mathcal{P}, Z, \check{p}r, P)$ and its associated vector bundle $(\mathcal{V}, Z, pr, \mathcal{V}_{(\mu)}, P)$ where the base space is a submanifold of the homogeneous coset space $Z := Sp(8, \mathbb{R})/P$. Recall that a point $g \in \mathcal{P} \equiv Sp(8, \mathbb{R})$ is an admissible map $g : \mathcal{V}_{(\mu)} \rightarrow \mathcal{V}$. Since we are stipulating unitary irreps of $U(4)$, there is a unique vacuum $\mathbf{v}_{w_-} \in \mathcal{V}_{(\mu)}$ invariant under P so that $g(\mathbf{v}_{w_-})$ can be identified with the zero-section in \mathcal{V} . It is important that the elements of \mathfrak{Z}_+ mutually commute since then $\exp_{\mathfrak{Z}_+}(\mathcal{V}_{(\mu)})$ induces a foliation of \mathcal{V} compatible with the fiber structure, i.e. the leaves are homeomorphic to Z .

step 6: For each relevant $\mathcal{V}_{(\mu)}^{(r)}$ (labeled by r), consider normalized equivariant, smooth p-forms $\check{\psi} \in C_C(\bigwedge^p T^*\mathcal{P}, \mathcal{V}_{(\mu)}^{(r)})$ with norm

$$\|\check{\psi}\|_{L^2} = \left(\text{tr} \int_{Sp(8, \mathbb{R})} \check{\psi} \wedge * \check{\psi} \right)^{1/2} \quad (2.12)$$

where the trace is with respect to the scalar product on $\mathcal{V}_{(\mu)}^{(r)}$. The induced unitary representations are defined by¹³

$$\text{UInd}_P^{Sp(8, \mathbb{R})^{(r)}} = \{ \check{\psi} \in L^2(\mathcal{P}, \mathcal{V}_{(\mu)}^{(r)}) \mid \check{\psi}(gp) = N(p) \bar{\varrho}(p^{-1}) \check{\psi}(g) \} \quad (2.13)$$

where $p \in P$, the normalization $N^2(p) := \Delta_P(p)/\Delta_{Sp(8, \mathbb{R})}(p)$ with modular function $\Delta_G(g) = |\det \text{Ad}_G(g)|$, and the continuous map $\bar{\varrho} : P \rightarrow L(\mathcal{V}_{(\mu)}^{(r)})$ is a unitary lowest-weight irrep.

step 7: Construct the Whitney sum bundle

$$\begin{aligned} \mathcal{W}_{\mathcal{V}} &:= \left(\bigoplus_r \mathcal{V}^{(r)}, Z, pr, \bigoplus_r \mathcal{V}_{(\mu)}^{(r)}, P \right) \\ &=: (\mathcal{W}, Z, pr, \mathcal{W}_{(\mu)}, P) \end{aligned} \quad (2.14)$$

¹²The extension is trivial because, as follows from the commutation relations, \mathfrak{Z}_- annihilates every element in $\mathcal{V}_{(\mu)}$.

¹³Strictly speaking, the literature defines induced representations for vector-valued 0-forms. The extension here to vector-valued p-forms is anticipating gauge-type state-vectors.

using all relevant unitary irrep modules $\mathcal{V}_{(\mu)}^{(r)}$. The typical fiber $\mathcal{W}_{(\mu)}$ is Hilbert. The induced urep module is

$$\text{UInd}_P^{Sp(8, \mathbb{R})} = \bigoplus_r \text{UInd}_P^{Sp(8, \mathbb{R})^{(r)}}. \quad (2.15)$$

The induced urep $\rho : Sp(8, \mathbb{R}) \rightarrow L(\text{UInd}_P^{Sp(8, \mathbb{R})})$, which will *not* be irreducible in general, can be expressed as

$$(\rho(g)\check{\psi})(g_o) = \check{\psi}(g^{-1}g_o) =: \check{\psi}_g(g_o) \quad (2.16)$$

where $g_o, g \in Sp(8, \mathbb{R})$.

2.1.3 Hilbert space

From the induced urep module $\text{UInd}_P^{Sp(8, \mathbb{R})}$, we want to construct a physical Hilbert space of state-vectors \mathcal{H}_Q . We restrict attention to a particular value of p ; remembering that the full Hilbert space will include all relevant p -forms.

Since the generators associated with the homogenous space Z mutually commute, they can be simultaneously diagonalized — meaning state-vectors of the quantum system can be parametrized by the *smooth manifold* Z . Also, we can transfer the furnishing space of the induced representation since $\check{\psi}$ and $\psi \in \Gamma(\bigwedge^p T^*Z, \mathcal{W})$ are related by $\check{\psi}(g) = g^{-1} \circ \psi(z)$ with $\Pi(g) = z = gz_0$ where z_0 is a choice of origin in Z . If a canonical local section σ_i on the principal bundle is chosen relative to a local trivialization $\{U_i, \varphi_i\}$, then $\check{\psi}$ and ψ are *canonically* related, and we can identify $\psi \equiv \sigma_i^* \check{\psi} \cdot [1]$

Since $\check{p}r(g\sigma_r(z)) = g\check{p}r(\sigma_r(z)) = gz$, then $g\sigma_i(z)$ must be a point in the fiber over gz , i.e. $g\sigma_i(z) = \sigma_i(gz)p$ for some $p \in P$. Hence, using canonical local sections relative to a given trivialization yields a canonical induced representation on $\Gamma(\bigwedge^p T^*Z, \mathcal{W})$;

$$\begin{aligned} (\rho(g)\psi)(z) &= (\rho(g)\check{\psi})(\sigma_i(z)) \\ &= \check{\psi}(g^{-1}\sigma_i(z)) \\ &= \bar{\varrho}(p^{-1})\sigma_i^*\check{\psi}(g^{-1}z) = \bar{\varrho}(p^{-1})\psi(g^{-1}z) \\ &=: \bar{\varrho}(p^{-1})\psi_g(z). \end{aligned} \quad (2.17)$$

The urep ρ can be used to define a gauge-invariant $*$ -homomorphism $\pi_z : L_B(\mathcal{H}) \rightarrow L_B(\mathcal{W}_z)$ by

$$(\pi_z(\rho(g))) \mathbf{v}_{w_{g_o}} := (\rho(g)\psi)(z) \quad (2.18)$$

where $(z, \mathbf{v}_{w_{g_o}})$ is the representative of $\psi(z)$ in a local trivialization.

There is ambiguity in this action associated with $\bar{\varrho}(p)$: It can be interpreted either as an arbitrary choice of local section σ_i or an arbitrary choice of basis on each fiber \mathcal{W}_x . Since a particular choice of either is not physically relevant, physical states should not depend on the choice. This implies that, if we want ψ to represent a

physical state, it should be covariant under the right action of P . But this is just the fiber bundle statement of gauge invariance. Conclude that a physical state is represented by an equivalence class $[\psi]$ with equivalence relation $\psi(z) \sim \psi(zp)$ for all $p \in P$ and $z \in U_i$.¹⁴

The condition that an entire equivalence class $[\psi] \in \mathcal{H}_Q$ represents a physical state is readily handled in the bundle framework; just insist that the p-forms used to define the induced representation are *horizontal*. In effect this simply means that p-forms $\check{\psi}$ on \mathcal{P} are defined in terms of an exterior covariant derivative D associated with a choice of connection on \mathcal{P} .

Evidently, the module $\mathcal{H} = L^2(\bigwedge^p T^*Z, \mathcal{W}) \subset \Gamma(\bigwedge^p T^*Z, \mathcal{W})$ furnishes a urep of $Sp(8, \mathbb{R})$ with subspace $\mathcal{H}_Q \subseteq \mathcal{H}$ spanned by $[\psi] \equiv \psi_{hor}$. Use the quasi-invariant measure μ_P on Z and the Hermitian inner product on $\mathcal{W}_{(\mu)}$ to construct a bundle metric on \mathcal{W} . Equip \mathcal{H} with the Hermitian inner product induced from \mathcal{W}_x (equivalently from the Haar measure on \mathcal{P})

$$\langle \psi_1 | \psi_2 \rangle_{\mathcal{H}} := \int_Z (\psi_1(z) | \psi_2(z))_{\mathcal{W}_z} d\mu_P(z) =: \int_Z \psi_1 \wedge * \psi_2. \quad (2.19)$$

and complete \mathcal{H} with respect to the associated norm. Then \mathcal{H} can be identified with the quantum Hilbert space. Accordingly, \mathcal{H}_Q represents the physical Hilbert space. Note that this induced urep is not irreducible in general.

Remark 2.2 *The induction game can be played on the intermediate vector bundle $(\mathcal{I}, Sp(8, \mathbb{R})/U(4), pr, \mathcal{W}_{(\mu)}, U(4))$ where $Sp(8, \mathbb{R})/U(4)$ can be interpreted as a phase space. Non-trivial cross sections will exist for vanishing Euler class, and a 2-form on the space of sections can be defined in the usual way in terms of the non-degenerate 2-form on the base space. However, since the elements of $\mathfrak{Z} := \mathfrak{Z}_+ \oplus \mathfrak{Z}_-$ do not all commute, $\exp \mathfrak{z}(\mathcal{W}_{(\mu)})$ does not yield a compatible foliation of \mathcal{I} . Consequently, the relationship between $\check{\psi}$ and ψ that allowed identification of the Hilbert space with the space of smooth cross sections no longer makes physical sense, i.e. generically more than one cross section is associated with an orbit of $\exp \mathfrak{z}$. To remedy the situation, it is enough to pick any ten commuting elements in \mathfrak{Z} ; thus defining a “polarization” on the phase space which in turn induces a compatible foliation.*

2.1.4 Coherent states

The structure of the coset space $Z := Sp(8, \mathbb{R})/P$ immediately suggests defining Perelomov-type CS. Many useful details regarding these and other types of CS can be found in [12] and [13].¹⁵

¹⁴It is important to keep in mind that the equivalence relation is strictly imposed only in a local trivialization $\{U_i, \varphi_i\}$ since it springs from ambiguity of the choice of local section — it is not meant to be a global statement.

¹⁵Another important and useful type of CS is the Barut/Girardello-type. They are analogs of momentum eigenstates of Poincaré.

The construction requires a complexified coset space $Z^{\mathbb{C}} := Sp(8, \mathbb{C})/P^{\mathbb{C}}$. Recall that $g \in G^{\mathbb{C}}$ can be viewed as an admissible map $g : \mathcal{W}_{(\mu)}^{\mathbb{C}} \rightarrow \mathcal{W}^{\mathbb{C}}$. Given a local trivialization $\{U_i, \varphi_i\}$ of the Whitney sum bundle and a local chart $\phi : U_i \rightarrow \mathbb{C}^{10}$, a point $w \in \pi^{-1}(U_i) \subset \mathcal{W}^{\mathbb{C}}$ can be represented on $\mathbb{C}^{10} \times \mathcal{W}_{(\mu)}^{\mathbb{C}}$ as

$$|\phi(z); \mu\rangle := \left(\exp \left\{ \frac{1}{2} \sum_{i,j} z_{ij}^* \mathbf{e}_{ij} \right\} \right) |\mu\rangle \quad (2.20)$$

where the point $z \in U_i$ has coordinates $\phi(z) = z_{ij}^* \in \mathbb{C}^{10}$, the vector $|\mu\rangle$ represents a basis of $\mathcal{W}_{(\mu)}^{\mathbb{C}}$, and we used the Mackey factorization $g = \exp\{\frac{1}{2} \sum_{i,j} z_{ij}^* \mathbf{e}_{ij}\} p$.

To simplify notation a bit write $|\phi(z); \mu\rangle \rightarrow |z^*; \mu\rangle$. Then a state-vector $\psi \in \mathcal{H}$ can be modeled locally on $U_i \times \mathcal{W}_{(\mu)}^{\mathbb{C}}$. Explicitly,

Definition 2.1 *Given a local trivialization of the bundle $\mathcal{W}_{\mathcal{M}}^{\mathbb{C}}$, the CS model of a state-vector $\psi \in \mathcal{H}$ is defined by¹⁶*

$$(z; \mu | \psi) =: \psi_{\mu}(z) \equiv \sigma_i^* \check{\psi}(z) \quad (2.21)$$

where σ_i is the canonical local section and $z \in Z_{\partial} \subseteq Z$. The space Z_{∂} is determined by boundary conditions on $\psi_{\mu}(z)$.

Similarly, the $*$ -homomorphism defined in (2.18) has a CS realization:

Definition 2.2 *The CS model of an operator $O \in L_B(\mathcal{H})$ is defined by*

$$(z; \mu | O \psi) =: \hat{O} \psi_{\mu}(z) . \quad (2.22)$$

We call $\psi_{\mu}(z)$ a coherent state wave function or coherent state for short. It is a column vector according to the relevant unitary irreps of $U(4)$ collectively labeled by $\mu = (\mu^{(r_1)}, \dots, \mu^{(r_n)})$. Since $\mathcal{V}_{w-}^{(r)}$ are unitary irreps, $\psi_{\mu}(z)$ is comprised of components $\psi_{\mu}(z) = (\psi_{\mu}^{r_1}(z), \dots, \psi_{\mu}^{r_n}(z))$ that do not mix — a kind-of super selection. We will often restrict to a specific component $\psi_{\mu}(z) = (z; \mu | \psi) \in \mathcal{V}_{(\mu)}^{(r)}$ for notational and conceptual simplicity.

2.1.5 Matrix CS

The isomorphism between the space of z_{ij} parameters and the vector space of symmetric 4×4 matrices allows to write the coherent state basis as

$$|z^*; \mu\rangle = |Z^*; \mu\rangle := \left(\exp \left\{ \frac{1}{2} \text{tr}(Z^* \mathfrak{E}_+) \right\} \right) |\mu\rangle \quad (2.23)$$

¹⁶This mixed bracket notation is a bit strange: On one hand it should not be confused with the Hilbert space inner product $\langle \cdot | \cdot \rangle_{\mathcal{H}}$ or the inner product $(\cdot | \cdot)_{\mathcal{W}_{(\mu)}}$. On the other hand, it emphasizes that the object it defines is a CS model of a state-vector. It must be kept in mind that the notation $(\cdot | \cdot)$ implicitly assumes a local trivialization, and can be interpreted as the expression of a state-vector in the “ $z \otimes \mu$ representation”.

where $\mathbf{Z}^* \in M_4^{sym}(\mathbb{C})$ is comprised of the coordinates z_{ij} and it is understood that U_i is modeled on $M_4^{sym}(\mathbb{C})$ — the space of symmetric 4×4 matrices with complex components.

To implement this, define the symmetric matrices \mathfrak{E}_+ and \mathfrak{E}_- with components $\{\mathfrak{e}_{ij}\}$ and $\{\mathfrak{e}_{ij}^*\}$ respectively, and \mathfrak{E}_U comprised of $\{\mathfrak{i}_{ij}\}$. Form the vector space $\mathcal{W}_{(\mu)}^{\mathbb{C}} := M_4^{sym}(\mathbb{C}) \otimes \mathcal{W}_{(\mu)}^{\mathbb{C}}$, and model Z on $M_4^{sym}(\mathbb{C})$. Given a chart on Z and a local trivialization on $\mathcal{W}^{\mathbb{C}}$, a point is represented by

$$\varphi_i(w) = |\mathbf{Z}^*; \boldsymbol{\mu}\rangle = \left(\exp \left\{ \frac{1}{2} \text{tr}(\mathbf{Z}^* \mathfrak{E}_+) \right\} \right) |\boldsymbol{\mu}\rangle . \quad (2.24)$$

Now define;

Definition 2.3 *A CS model of a state-vector in the matrix picture is defined by*

$$(\mathbf{Z}; \boldsymbol{\mu} | \psi\rangle := (\boldsymbol{\mu} | \left(e^{\frac{1}{2} \text{tr}(\mathbf{Z} \mathfrak{E}_-)} \right) \psi\rangle = \boldsymbol{\psi}_{\boldsymbol{\mu}}(\mathbf{Z}) , \quad (2.25)$$

and the model of an operator (not necessarily bounded) is

$$(\mathbf{Z}; \boldsymbol{\mu} | O \psi\rangle := \widehat{O} \boldsymbol{\psi}_{\boldsymbol{\mu}}(\mathbf{Z}) . \quad (2.26)$$

Referring to [12],¹⁷ an explicit CS realization of the Lie algebra generators for 0-forms in a local trivialization $U_i \times \mathcal{W}_{(\mu)}^{\mathbb{C}}$ in the matrix picture is given by

$$\widehat{\mathfrak{E}}_- = \begin{pmatrix} 2 \frac{\partial}{\partial z^1} & \frac{\partial}{\partial z^{12}} & \frac{\partial}{\partial z^{13}} & \frac{\partial}{\partial z^{14}} \\ \frac{\partial}{\partial z^{12}} & 2 \frac{\partial}{\partial z^2} & \frac{\partial}{\partial z^{23}} & \frac{\partial}{\partial z^{24}} \\ \frac{\partial}{\partial z^{13}} & \frac{\partial}{\partial z^{23}} & 2 \frac{\partial}{\partial z^3} & \frac{\partial}{\partial z^{34}} \\ \frac{\partial}{\partial z^{14}} & \frac{\partial}{\partial z^{24}} & \frac{\partial}{\partial z^{34}} & 2 \frac{\partial}{\partial z^4} \end{pmatrix} \otimes \mathbb{I} =: \partial_{\mathbf{Z}} \otimes \mathbb{I} , \quad (2.27)$$

$$\begin{aligned} \widehat{\mathfrak{E}}_+ &= [\mathbf{Z} \partial_{\mathbf{Z}} - (d+1)] \mathbf{Z} \otimes \mathbb{I} + \mathbf{Z} \otimes \mathbb{U} + (\mathbf{Z}^T \otimes \mathbb{U})^T \\ &= [\mathbf{Z} \partial_{\mathbf{Z}} - (d+1)] \mathbf{Z} \otimes \mathbb{I} + \mathbf{Z} \otimes \mathbb{U} + (\mathbf{Z} \otimes \mathbb{U})^T \\ &=: [\mathbf{Z} \partial_{\mathbf{Z}} - (d+1)] \mathbf{Z} \otimes \mathbb{I} + \text{Sym}(\mathbf{Z} \otimes \mathbb{U}) , \end{aligned} \quad (2.28)$$

and

$$\widehat{\mathfrak{E}}_U = \mathbf{Z} \partial_{\mathbf{Z}} \otimes \mathbb{I} + \widehat{Id} \otimes \mathbb{U} \quad (2.29)$$

where

$$(\mathbb{U})_{\boldsymbol{\mu}' \boldsymbol{\mu}} := (\boldsymbol{\mu}' | \varrho'(\mathfrak{E}_U) | \boldsymbol{\mu})_{\mathcal{W}_{(\mu)}^{\mathbb{C}}} \quad (2.30)$$

¹⁷Note that our notation differs a bit from [12]: we put $\mu := \lambda + n/2$ where λ is the lowest weight characterizing the $U(4)$ unitary irrep and $n \geq 2d = 8$.

with \mathfrak{E}_U the generators of $U(4)$. These do not belong to $L_B(\mathcal{H})$ but their unitary exponentials do.

In words, the set of matrix-valued operators $\{\widehat{\mathfrak{E}}_+, \widehat{\mathfrak{E}}_-, \widehat{\mathfrak{E}}_U\}$ is a Perelomov-type CS model of $\{\mathfrak{e}_{ij}, \mathfrak{e}_{ij}^*, \mathfrak{i}_{ij}\}$ in the matrix picture, and unitary exponentiation realizes an induced urep of $Sp(8, \mathbb{C})$.

The overlap kernel for $(z', z) \in U_i$ has been calculated explicitly in [12];

$$(\mathbf{Z}'; \boldsymbol{\mu}' | \mathbf{Z}^*; \boldsymbol{\mu}) = (\boldsymbol{\mu}' | \rho(e^{\text{tr} B \mathfrak{E}_U}) | \boldsymbol{\mu})_{\mathcal{W}_{(\boldsymbol{\mu})}^{\mathbb{C}}} =: (\mathbb{K}^{-1}(\mathbf{Z}', \mathbf{Z}^*))_{\boldsymbol{\mu}' \boldsymbol{\mu}} \quad (2.31)$$

where $e^{-B} = (Id - \mathbf{Z}' \mathbf{Z}^*)$. The associated resolution of the identity is

$$\widehat{Id} = \int_{U_i} |\mathbf{Z}^*; \boldsymbol{\mu}\rangle \mathbf{d}\boldsymbol{\sigma}(z) \langle \mathbf{Z}; \boldsymbol{\mu}| \quad (2.32)$$

where

$$\mathbf{d}\boldsymbol{\sigma}(z) := \mathcal{N} \frac{\mathbb{K}(\mathbf{Z}, \mathbf{Z}^*)}{\det(Id - \mathbf{Z} \mathbf{Z}^*)^{(4+1)}} dz =: \mathbb{P}(z) dz. \quad (2.33)$$

\mathcal{N} is a normalization constant and $\widehat{Id} \boldsymbol{\psi}_{\boldsymbol{\mu}}(\mathbf{Z}) = (Id \boldsymbol{\psi})(\mathbf{Z})$ with Id the identity operator on \mathcal{H} .

From these, one obtains the local CS superposition on $\mathcal{W}^{\mathbb{C}}$;

$$|\psi\rangle_i = \int_{U_i} |\mathbf{Z}^*; \boldsymbol{\mu}\rangle \mathbf{d}\boldsymbol{\sigma}(z) \langle \mathbf{Z}; \boldsymbol{\mu} | \psi \rangle \quad (2.34)$$

which must then be extended globally to $Z^{\mathbb{C}} \setminus S^n$ with S^n the unit sphere in $Z^{\mathbb{C}}$ and Dirichlet/Neumann boundary conditions on the sphere $(\mathbf{Z}; \boldsymbol{\mu} | \psi)|_{S^n} = \Psi_{\boldsymbol{\mu}}$. Similarly, assuming $\Psi_{\boldsymbol{\mu}} = 0$ for simplicity,

$$\begin{aligned} \langle \psi | O \psi \rangle_{\mathcal{H}} &= \int_{Z^{\mathbb{C}} \setminus S^n} \boldsymbol{\psi}_{\boldsymbol{\mu}'}^{\dagger}(\mathbf{Z}') \mathbb{P}(z') \widehat{O} \boldsymbol{\psi}_{\boldsymbol{\mu}}(\mathbf{Z}') dz' \\ &= \int_{Z^{\mathbb{C}} \setminus S^n} \int_{Z^{\mathbb{C}} \setminus S^n} \boldsymbol{\psi}_{\boldsymbol{\mu}'}^{\dagger}(\mathbf{Z}') \mathbb{P}(z') (\mathbf{Z}'; \boldsymbol{\mu}' | O | \mathbf{Z}^*; \boldsymbol{\mu}) \mathbb{P}(z) \boldsymbol{\psi}_{\boldsymbol{\mu}}(\mathbf{Z}) dz' dz \\ &=: \int_{Z^{\mathbb{C}} \setminus S^n} \int_{Z^{\mathbb{C}} \setminus S^n} \boldsymbol{\psi}_{\boldsymbol{\mu}'}^{\dagger}(\mathbf{Z}') \mathbb{K}_O^{-1}(\mathbf{Z}', \mathbf{Z}^*) \boldsymbol{\psi}_{\boldsymbol{\mu}}(\mathbf{Z}) dz dz'. \end{aligned} \quad (2.35)$$

With proper restrictions $\widehat{O} \boldsymbol{\psi}_{\boldsymbol{\mu}'}(\mathbf{Z}') = (\mathbf{Z}'; \boldsymbol{\mu}' | O \psi)$ can be rendered a distribution, and $\mathbb{K}_O(\mathbf{Z}', \mathbf{Z}^*)$ can be interpreted as the CS model of the Green's function associated with operator O .

Remark 2.3 *The definition of CS in terms of the map $\exp : \mathfrak{Sp}(8) \rightarrow Sp(8, \mathbb{R})$ is problematic because, although it makes sense in a local neighborhood of $e \in Sp(8, \mathbb{R})$, it is not one-to-one or onto ([27] pg. 18). A more sound approach is to define Cayley CS by $|\mathbf{Z}^*; \boldsymbol{\mu}\rangle := \mathcal{C}(\mathbf{Z}^* \mathfrak{E}_+) | \boldsymbol{\mu}\rangle$ and conjugate CS by $|\mathbf{Z}; \boldsymbol{\mu}\rangle := \mathcal{C}^{\dagger}(\mathbf{Z} \mathfrak{E}_-) | \boldsymbol{\mu}\rangle$ where \mathcal{C}*

denotes the Cayley transform $\mathcal{C}(\mathbf{Z}^* \mathfrak{E}_+) = (Id + \mathbf{Z}^* \mathfrak{E}_+)(Id - \mathbf{Z}^* \mathfrak{E}_+)^{-1}$ and $\mathbf{Z} \mathbf{Z}^* \neq 1$. Then $|\mathbf{Z}^*; \boldsymbol{\mu}\rangle \oplus |\mathbf{Z}; \boldsymbol{\mu}\rangle$ along with J is a symplectic vector space, and $g \in Sp(8, \mathbb{R})$ acts by $\exp\{\pm \mathfrak{g}\}$ on each component subspace. ([27] pg. 18) So the original CS model remains intact, but there are two copies — a conjugate pair. This is just a generalization of the Bargmann representation in elementary QM.

2.1.6 CS vacuum

Let $\mathbf{w}_- = (w_-^{(r_1)}, \dots, w_-^{(r_n)})$ denote the collection of dominant-integral lowest weights. We define the ground state by $\check{\psi}_0(g) := \mathbf{v}_{\mathbf{w}_-} \in \mathcal{W}_{(\boldsymbol{\mu})}^{\mathbb{C}} \forall g \in Sp(8, \mathbb{R})$. And we define a vacuum-state to be the ground state of a unitary irrep ρ induced from the trivial partition $\boldsymbol{\mu} = [\mu, \mu, \mu, \mu]$. In this case, $\mathcal{W}_{(\boldsymbol{\mu})}^{\mathbb{C}} \equiv \mathcal{V}_1^{(0)}$ is irreducible and one-dimensional such that $(\rho(g)\check{\psi}_0)(g) \propto \mathbf{v}_{\boldsymbol{\mu}}$ for all $g \in Sp(8, \mathbb{R})$.

According to the definition, the CS model of the algebra generators acting on a ground state ψ_0 give $\widehat{\mathfrak{E}}_- \psi_0 = 0$, $\widehat{\mathfrak{E}}_+ \psi_0 = \mathbf{Z} \otimes (\mathbb{I} + Sym(\mathbb{U})) \psi_0$, and $\widehat{\mathfrak{E}}_U \psi_0 = Id \otimes \mathbb{U} \psi_0$. Obviously the vacuum-state is covariant under $\widehat{\mathfrak{E}}_-$ and $\widehat{\mathfrak{E}}_U$ as are $U(4)$ degenerate ground states if they exist.

It is appropriate to call P a gauge group and $\mathbf{v}_{\boldsymbol{\mu}}$ a gauge-invariant vacuum. This suggests a natural definition of the CS model of a vacuum state-vector;

Definition 2.4 The CS model of a vacuum state-vector $\varphi_0 \in \mathcal{H}$ is defined by

$$(z; \boldsymbol{\mu} | \varphi_0) := \mathbf{v}_{\boldsymbol{\mu}}(z) \equiv \mathbf{v}_{\boldsymbol{\mu}} \quad \forall z \in Z^{\mathbb{C}} \quad (2.36)$$

such that $\langle \varphi_0 | \varphi_0 \rangle_{\mathcal{H}} = |\mathbf{v}_{\boldsymbol{\mu}}|$.

Proposition 2.1 The VEV w.r.t. $\mathbf{v}_{\boldsymbol{\mu}}$ of any observable O is gauge invariant. Conversely, gauge-invariant observables induce degenerate ground states that span $\mathcal{W}_{(\boldsymbol{\mu})}^{\mathbb{C}}$.

Proof: By definition, the vacuum furnishes the trivial one-dimensional representation of the parabolic subgroup P so $\check{\varphi}_0(gp^{-1}) = N(p)\bar{\varrho}(p)\check{\varphi}_0(g) = N(p)\mathbf{v}_{\boldsymbol{\mu}} \forall g \in Sp(8, \mathbb{R})$ and $p \in P$. Hence, the VEV with respect to $\mathbf{v}_{\boldsymbol{\mu}}$ of any observable $O \in \mathfrak{A}$ is automatically gauge invariant;

$$\langle \varphi_0 | O \varphi_0 \rangle_{\mathcal{H}} = \langle \rho(p)\varphi_0 | O \rho(p)\varphi_0 \rangle_{\mathcal{H}} = \langle \varphi_0 | \rho(p)^* O \rho(p) \varphi_0 \rangle_{\mathcal{H}} \quad (2.37)$$

where $O \in L(\mathcal{H})$ is the (not necessarily bounded) linear operator representing the observable $O \in \mathfrak{A}$. Equivalently, if the trace exists,

$$\text{tr } \mathbb{K}_O^{-1}(\mathbf{Z}', \mathbf{Z}^*) = \text{tr } \rho^{-1}(p) \mathbb{K}_O^{-1}(\mathbf{Z}', \mathbf{Z}^*) \rho(p) = \text{tr } Ad(p) \mathbb{K}_O^{-1}(\mathbf{Z}', \mathbf{Z}^*) . \quad (2.38)$$

Similarly, if an observable is gauge invariant, then $\langle \psi_{gp} | O \psi_{gp} \rangle_{\mathcal{H}} = \langle \psi_g | O \psi_g \rangle_{\mathcal{H}}$. In particular, $\langle \psi_p | O \psi_p \rangle_{\mathcal{H}} = \langle \psi_0 | O \psi_0 \rangle_{\mathcal{H}}$. Roughly speaking, the entire module $\mathcal{W}_{(\boldsymbol{\mu})}^{\mathbb{C}}$ is spanned by gauge-degenerate ground states — relative to gauge-invariant observables. In other words, gauge-invariant observables cannot distinguish elements of $\mathcal{W}_{(\boldsymbol{\mu})}^{\mathbb{C}}$. \square

Remark 2.4 *Let's take stock of what we have so far. We start with a parabolic decomposition of the symplectic group $Sp(8, \mathbb{R})$ and use this decomposition to construct induced representations. From the induced representations, the Hilbert space of state-vectors $L^2(\bigwedge^p T^*Z, \mathcal{W})$ or $L^2(\bigwedge^p T^*\mathcal{P}, \mathcal{W}_{(\mu)})$ is constructed. Complexifying Z allows the construction of matrix CS, allowing to model ψ by the ‘wave function’ $\psi_\mu(\mathbf{Z})$.¹⁸ The wave-function model yields explicit realizations of operators in a manner familiar from elementary QM: In particular we indicated expressions for the operators associated with the generators of $\mathfrak{Sp}(8, \mathbb{C})$ obtained by [12]. Just like elementary QM, the wave function $\psi_\mu(\mathbf{Z})$ can be interpreted as the state-vector ψ in the ‘CS representation’ $|\mathbf{Z}^*; \mu\rangle$.*

What remains is to construct a model of \mathfrak{A} and represent the dynamics. As discussed previously, both of these spring from the assumed dynamical group $Sp(8, \mathbb{R})$.

2.2 The C^* algebra

Much of the material in this section was presented in [1], but it is included here in abridged form for convenient reference.

Since $Sp(8, \mathbb{R})$ is supposed to be the ‘shadow’ of the topological group of units G of the quantum algebra \mathfrak{A} , it is reasonable to assume that \mathfrak{A} can be modeled by functionals $F : Sp(8, \mathbb{R}) \rightarrow L_B(\mathcal{H})$. After all, we expect the bracket structure of G to be carried by G_λ . Therefore, the space of functionals comprised of $F : Sp(8, \mathbb{R}) \rightarrow L_B(\mathcal{H})$, which is a C^* -algebra when properly defined, should be a good model of \mathfrak{A} — in the sense that it contains any observable that one could hope to measure.

We use functional Mellin transforms to construct the C^* -algebra. Functional Mellin transforms are a particular type of functional integral defined in [30]. Roughly, a functional integral is defined by a *family* of integral operators $\text{int}_\Lambda : \mathbf{F}(G_\Lambda) \rightarrow \mathfrak{B}$ where \mathfrak{B} is a Banach algebra and $\mathbf{F}(G_\Lambda)$ is a family of spaces of integrable functions $f \in L^1(G_\lambda, \mathfrak{B})$ for all $\lambda \in \Lambda$. For the purposes of this paper, $G_\lambda = Sp(8, \mathbb{R})$ and $\mathfrak{B} \equiv L_B(\mathcal{H})$.

To define functional Mellin transforms in the context of SQM, start with the data $(G, L_B(\mathcal{H}), G_\Lambda^\mathbb{C})$ where $G_\Lambda^\mathbb{C} = Sp(8, \mathbb{C})$ and \mathcal{H} is a urep module.

Definition 2.5 ([15]) *Let the map $\rho : G_{A,\lambda}^\mathbb{C} \rightarrow L_B(\mathcal{H})$ be a continuous, injective homomorphism, and $\pi_z : L_B(\mathcal{H}) \rightarrow L_B(\mathcal{W}_z)$ be the non-degenerate $*$ -homomorphism defined in (2.18). Define continuous functionals $\mathbf{F}(G_A^\mathbb{C}) \ni F : G_A^\mathbb{C} \rightarrow L_B(\mathcal{H})$ equivariant under right-translations¹⁹ according to $F(gh) = F(g)\rho(h)$. Then the functional Mellin transform $\mathcal{M}_\lambda : \mathbf{F}(G_A^\mathbb{C}) \rightarrow L_B(\mathcal{H})$ is defined by*

$$\mathcal{M}_\lambda [F; \alpha] := \int_{G_A^\mathbb{C}} F(gg^\alpha) \mathcal{D}_\lambda g = \int_{G_A^\mathbb{C}} F(g)\rho(g^\alpha) \mathcal{D}_\lambda g \quad (2.39)$$

¹⁸This of course is a very loose term since $\psi_\mu(\mathbf{Z})$ is generally a p-form.

¹⁹This prescription is for left-invariant Haar measures. For right-invariant Haar measures impose equivariance under left-translations.

where $\alpha \in \mathbb{S} \subset \mathbb{C}$, $g^\alpha := \exp_G(\alpha \log_G g)$ and $\pi_z(F(g)\rho(g^\alpha)) \in L_B(\mathcal{W}_z)$ where the space of bounded linear operators $L_B(\mathcal{W}_z)$ is given the operator-norm topology. Denote the space of Mellin integrable functionals by $\mathbf{F}_\mathbb{S}(G_A^\mathbb{C})$.

Define a norm on $\mathbf{F}_\mathbb{S}(G_A^\mathbb{C})$ by $\|F\| := \sup_\alpha \|F\|_\alpha$ where

$$\|F\|_\alpha := \sup_\lambda \|\mathcal{M}[f(g_\lambda); \alpha]\| < \infty, \quad \alpha \in \mathbb{S}. \quad (2.40)$$

Assume that $\mathbf{F}_\mathbb{S}(G_A^\mathbb{C})$ can be completed with respect to $\|F\|$ or some other suitably defined norm. Then

Proposition 2.2 ([15] prop. 4.2) *$\mathbf{F}_\mathbb{S}(G_A^\mathbb{C})$ is a C^* -algebra such that $\|F^*\|_\alpha = \|F\|_\alpha$ when endowed with an involution defined by $F^*(g^{1+\alpha}) := F(g^{-1-\alpha})^* \Delta(g^{-1})$ and suitable topology.*

Our postulate is that the space of Mellin integrable functionals $\mathbf{F}_\mathbb{S}(G_A^\mathbb{C})$ models the C^* -algebra \mathfrak{A} that characterizes the physical properties of a quantum system. Since $L_B(\mathcal{H})$ is non-commutative and $Sp(8, \mathbb{R})$ is non-abelian, it follows from Prop. 4.6 [15] that $\mathcal{M}_\lambda[\cdot; 1] =: \mathcal{R}_\lambda^{(1)} : \mathbf{F}_\mathbb{S}(G_A^\mathbb{C}) \rightarrow L_B(\mathcal{H})$ is a $*$ -representation. Remark that $\mathcal{R}_\lambda^{(1)}$ is just a crossed product [14]. However, it is still useful to maintain the general Mellin transform setup because $\mathcal{M}_\lambda[\cdot; \alpha]$ will be a $*$ -representation for all $\alpha \in \mathbb{S}$ when one wants to calculate functional traces and functional determinants.

An important property of $\mathcal{R}_\lambda^{(1)}(\mathbf{F}_\mathbb{S}(G_A^\mathbb{C}))$ necessary for the consistency of SQM is that it contains a copy of $\rho(Sp(8, \mathbb{R}))$ (which can only happen for non-abelian $G_A^\mathbb{C}$ if $\alpha = 1$ and ρ is unitary);

Proposition 2.3 ([14] prop. 2.34) *Let $U(\mathcal{R}_\lambda^{(1)}(\mathbf{F}_\mathbb{S}(G_A^\mathbb{C})))$ denote the unitary group of units of $\mathcal{R}_\lambda^{(1)}(\mathbf{F}_\mathbb{S}(G_A^\mathbb{C}))$. Define $i_h : \mathbf{F}_\mathbb{S}(G_A^\mathbb{C}) \rightarrow \mathbf{F}_\mathbb{S}(G_A^\mathbb{C})$ by $(i_h(F))(g) := \text{Ad}(h)F(gh)$ with $h, g \in G_A^\mathbb{C}$. Then the map $i_\lambda : G_{A,\lambda}^\mathbb{C} \rightarrow U(\mathcal{R}_\lambda^{(1)}(\mathbf{F}_\mathbb{S}(G_A^\mathbb{C})))$ by $h \mapsto (\mathcal{R}_\lambda^{(1)} \circ i_h)$ is an injective strictly continuous unitary valued homomorphism.*

Proof: The proof is left to the reference since $\mathcal{R}_\lambda^{(1)}(F) = \mathcal{M}_\lambda[F; 1]$ is just a crossed product. But note that ρ is unitary since $\alpha = 1$ and

$$\begin{aligned} i_\lambda(h)(F) = \mathcal{R}_\lambda^{(1)}(i_h(F)) &= \int_{G_A^\mathbb{C}} \rho(h)F(gg^\alpha h)\rho(h^{-1}) \mathcal{D}_\lambda g \Big|_{\alpha=1} \\ &= \int_{G_A^\mathbb{C}} \rho(h)F(gg^\alpha)\rho(h)\rho(h^{-1}) \mathcal{D}_\lambda g \Big|_{\alpha=1} \\ &= \mathcal{R}_\lambda^{(1)}(\rho(h)F) = \rho(h)\mathcal{R}_\lambda^{(1)}(F). \end{aligned} \quad (2.41)$$

In particular, $\mathcal{R}_\lambda^{(1)}(i_h(\text{Id})) = \rho(h)\mathcal{R}_\lambda^{(1)}(\text{Id}) = \rho(h)$. So despite appearances, our map i_h coincides with $i_G(h)$ of [14]. They are defined differently because we use equivariant functions F . \square

We already knew $\rho(G_A^{\mathbb{C}}) \subset L_B(\mathcal{H})$, but now we also have $\rho(G_A^{\mathbb{C}}) \subset U(\mathcal{R}_\lambda^{(1)}(\mathbf{F}_S(G_A^{\mathbb{C}})))$. Essentially, $G_A^{\mathbb{C}} \subset \mathbf{F}_S(G_A^{\mathbb{C}})$. This means that group elements are indeed observables as required, and $\mathcal{R}_\lambda^{(1)}(\mathbf{F}_S(G_A^{\mathbb{C}}))$ contains its associated operators.

The matrix CS model of $\mathcal{R}_\lambda^{(1)}(F)$ is $(\mathbf{Z}; \boldsymbol{\mu} | \mathcal{R}_\lambda^{(1)}(F) \psi) = \widehat{\mathcal{R}_\lambda^{(1)}(F)} \psi_\mu(\mathbf{Z})$. In particular, for F of the form $F = E^{-iH(t)}$ with self-adjoint $H(t)$, this becomes

$$(\mathbf{Z}; \boldsymbol{\mu} | \mathcal{R}_\lambda^{(1)}(F) \psi) = e^{-f(\widehat{\mathfrak{G}_\lambda^{\mathbb{C}}})} \psi_\mu(\mathbf{Z}) \quad (2.42)$$

where $f(\widehat{\mathfrak{G}_\lambda^{\mathbb{C}}}) = \widehat{\mathcal{R}_\lambda^{(1)}(\text{Log } F^*)} = \widehat{\mathcal{R}_\lambda^{(1)}(iH(t))} =: iH_\lambda(t)$. ([1] sec. 3) The associated CS realization of the expectation is

$$\begin{aligned} \langle \psi | \mathcal{R}_\lambda^{(1)}(F) \psi \rangle_\lambda &= \int_{Z^{\mathbb{C}} \setminus S^n} \psi_{\mu'}^\dagger(\mathbf{Z}') \mathbb{P}(z') \widehat{\mathcal{R}_\lambda^{(1)}(F)} \psi_{\mu'}(\mathbf{Z}') dz' \\ &=: \int_{Z^{\mathbb{C}} \setminus S^n} \int_{Z^{\mathbb{C}} \setminus S^n} \psi_{\mu'}^\dagger(\mathbf{Z}') \mathbb{K}_F^{-1}(\mathbf{Z}', \mathbf{Z}^*) \psi_\mu(\mathbf{Z}) dz dz' . \end{aligned} \quad (2.43)$$

Gauge-invariant observables are characterized by $O = \text{Ad}(p)O$ for all $p \in P$. The adjoint action on $\mathbf{F}_S(G_A^{\mathbb{C}})$ gets represented as an adjoint action $\text{Ad}(p)$ on $\mathcal{R}_\lambda^{(1)}(\mathbf{F}_S(G_A^{\mathbb{C}}))$ so that gauge-invariant operators obey $O_\lambda^{-\alpha} = \text{Ad}(p)O_\lambda^{-\alpha} = \rho(p)O_\lambda^{-\alpha}\rho(p^{-1})$. As an example of a gauge-invariant operator, suppose an observable F_P is left-equivariant (in addition to being right-equivariant) and a central function with respect to P , i.e. $F_P(pgp^{-1}) = F_P(g)$ for all $p \in P$. It follows from the definition of functional Mellin that $(F_P)_\lambda^{-\alpha} = \text{Ad}(p)(F_P)_\lambda^{-\alpha} = \rho(p)(F_P)_\lambda^{-\alpha}\rho(p^{-1})$. At least one class of this type is not too hard to construct: let $F_P(g) = E^{-\gamma_i \mathfrak{g}_U^i}$ with $\gamma_i \in \mathbb{C}$ and $\mathfrak{g}_U^i \in U(\mathfrak{G}_A^{\mathbb{C}})$ where $U(\mathfrak{G}_A^{\mathbb{C}})$ is the universal enveloping algebra. Then $F_P(g)$ is clearly left- and right-equivariant, and it will be a central function if $[\gamma_i \mathfrak{g}_U^i, \mathfrak{p}] = 0$ for all $\mathfrak{p} \in \mathfrak{P}$.

2.3 Automorphisms

2.3.1 Complex structures

By definition, $Sp(8, \mathbb{R})$ contains an *inner* automorphism j that is anti-involutive $j^2 = -e$ (with e the identity in $Sp(8, \mathbb{R})$) and satisfies $g^\dagger j g = j$ for all $g \in Sp(8, \mathbb{R})$. At the algebra level this relation becomes $\mathfrak{g}^\dagger j = -j \mathfrak{g}$. Evidently j induces an adjoint action $\text{Ad}(j) : \mathfrak{Sp}(8)_\pm \rightarrow \mathfrak{Sp}(8)_\mp$ by $\mathfrak{g}_\pm \mapsto j \mathfrak{g}_\pm j = \mathfrak{g}_\pm^\dagger = \mathfrak{g}_\mp$.

Having eigenvalues $\pm i$, the ‘complex structure’ j allows $\mathfrak{Sp}(8, \mathbb{R})$ to be given the structure of the complex algebra $\mathfrak{Sp}(8, \mathbb{C})$. This complex structure obviously extends to \mathcal{V}_μ via ϱ' . Consequently, for any complexified $\mathcal{V}_{(\mu)}^{\mathbb{C}}$, there exists a basis that diagonalizes $J := \rho(j)$ such that $\mathcal{V}_{(\mu)}^{\mathbb{C}} = \mathcal{V}_{(\mu)}^+ \oplus \mathcal{V}_{(\mu)}^-$ where

$$\mathcal{V}_{(\mu)}^\pm := \{ \mathbf{v} \in \mathcal{V}_{(\mu)}^{\mathbb{C}} \mid J \mathbf{v} = \pm i \mathbf{v} \} , \quad \forall \mathbf{v} \in \mathcal{V}_{(\mu)}^{\mathbb{C}} . \quad (2.44)$$

Hence, J provides a means to transfer objects formulated in the context of $Sp(8, \mathbb{C})$ into objects relevant to $Sp(8, \mathbb{R})$ and vice versa.

The automorphism j serves another important purpose. Recall that $\mathfrak{Sp}(8, \mathbb{R})$ is endowed with a non-degenerate, bi-linear, symmetric form B — the Cartan-Killing metric. Together with j , this defines a symplectic form by $\Omega(\cdot, \cdot) := B(\cdot, \text{Ad}(j)\cdot)$. So $\mathfrak{Sp}(8, \mathbb{R})$ has the structure of a symplectic vector space. Moreover, the metric and symplectic structures on $\mathfrak{Sp}(8, \mathbb{R})$ can be combined to construct an hermitian inner product $h : \mathfrak{Sp}(8, \mathbb{R}) \times \mathfrak{Sp}(8, \mathbb{R}) \rightarrow \mathbb{C}$ by

$$h(\mathfrak{g}_1, \mathfrak{g}_2) = B(\mathfrak{g}_1, \mathfrak{g}_2) - i\Omega(\mathfrak{g}_1, \mathfrak{g}_2), \quad \forall \mathfrak{g}_1, \mathfrak{g}_2 \in \mathfrak{Sp}(8, \mathbb{R}). \quad (2.45)$$

It is evident that h is just the hermitian inner product on $\mathfrak{Sp}(8, \mathbb{C})$ restricted to a real subspace $\mathfrak{R} \subset \mathfrak{Sp}(8, \mathbb{C})$ defined by $\mathfrak{R} \cap j\mathfrak{R} = \{0\}$.

In addition to the inner complex structure j , there are two *outer* anti-involutions k and l that satisfy $gk = kg$ and $gl = lg$ for all $g \in Sp(8, \mathbb{C})$ with $k^2 = -e$ and $l^2 = -e$. The four maps e, k, j, l exhaust all (anti)involutive automorphisms of $Sp(8, \mathbb{C})$. They are very special automorphisms because they endow $\mathfrak{Sp}(8, \mathbb{C})$ with three independent complex structures. Moreover, $jk = -kj$, $jl = -lj$, $kl = -lk$, and $jkl = -e$. So the linear maps $\{\text{Ad}(e), \text{Ad}(j), \text{Ad}(k), \text{Ad}(l)\}$ generate a quaternion algebra that acts on $\mathfrak{Sp}(8, \mathbb{C})$. However, none of them generates evolution: e trivially commutes with everything, j is not self-adjoint, and k, l are outer automorphisms.

Nevertheless, we can interpret their actions on $\mathcal{V}_{(\mu)}^{(r)}$ and hence \mathcal{H} . We have already seen that j exchanges $\mathfrak{g}_{\pm} \rightleftharpoons \mathfrak{g}_{\mp}$. Since we identify \pm with charges in the adjoint representation, $\text{Ad}(j)$ is interpreted as ‘particle \rightleftharpoons anti-particle’²⁰. To interpret k and l , construct an explicit representation; for example the defining representation. Then j, k, l can be represented by

$$J := \rho(j) = \begin{pmatrix} 0 & \mathbb{I}_{4 \times 4} \\ -\mathbb{I}_{4 \times 4} & 0 \end{pmatrix}, \quad (2.46)$$

$$K := \rho(k) = i \begin{pmatrix} 0 & \mathbb{I}_{4 \times 4} \\ \mathbb{I}_{4 \times 4} & 0 \end{pmatrix}, \quad (2.47)$$

and

$$L := \rho(l) = i \begin{pmatrix} \mathbb{I}_{4 \times 4} & 0 \\ 0 & -\mathbb{I}_{4 \times 4} \end{pmatrix}. \quad (2.48)$$

Evidently $\text{Ad}(k)$ can be interpreted as ‘particle \rightleftharpoons anti-particle’ without charge exchange. Finally, $\rho(l)$ affects the sign of the eigenvalues of stationary states. This suggests we interpret $\text{Ad}(l)$ as charge exchange with respect to evolution reversal. Finally, We have already insisted that \mathcal{H} is invariant under $\rho(j)$ because the highest and lowest weights were identified, but there is no reason to impose invariance under $\rho(k)$ and/or $\rho(l)$. On the other hand, invariance under $\rho(jkl) = -Id$ is assured and obviously resembles CPT-type symmetry.

²⁰The term ‘particle’ is being used loosely here. It refers to the excitation in the Fock space S_F associated with the creation/annihilation operators realizing $\mathfrak{Sp}(8)$.

2.3.2 Evolution

According to [1], quantum dynamics is generated by a continuous, unitary inner automorphism $F \mapsto \text{Ad}(h(t))F$ where $h(\mathbb{R}) \subset G_A^{\mathbb{C}}$ is a unitary subgroup. Here we suppose that $h(t)$ is determined by

$$\frac{dh(t)}{dt} := -i\mathfrak{h}_U(h(t)) = -i\mathfrak{h}_U(t)h(t), \quad h(t_a) = e, \quad t \in [t_a, t_b] \subseteq \mathbb{R} \quad (2.49)$$

where $\mathfrak{h}_U(t) \in U(\mathfrak{G}_A^{\mathbb{C}})$ is self-adjoint, $U(\mathfrak{G}_A^{\mathbb{C}})$ is the universal enveloping Lie algebra, and e is the identity group element.

Suppose that

$$\mathfrak{h}_U(t) = \sum_i \alpha_i(t) \mathfrak{g}_U^i \quad (2.50)$$

where $\alpha_i(t)$ are real analytic functions and $\mathfrak{g}_U^i \in U(\mathfrak{G}_A^{\mathbb{C}})$ are self-adjoint. Then according to Magnus' theorem, $h(t)$ can be written (for suitable t)

$$h(t) = e^{-i\tilde{\mathfrak{h}}(t)} := e^{-i\sum_i \beta_i(t) \mathfrak{g}_U^i} \quad (2.51)$$

where $\tilde{\mathfrak{h}}(t)$ is determined by

$$\frac{d\tilde{\mathfrak{h}}_U(t)}{dt} = \sum_{n=0}^{\infty} \frac{B_n}{n!} \text{ad}^n(\mathfrak{h}_U(t)) \tilde{\mathfrak{h}}_U(t) \quad (2.52)$$

where B_n are Bernoulli numbers and the map $\text{ad}^n(\mathfrak{h}_U(t))$ is defined recursively by $\text{ad}^0(\mathfrak{h}_U(t)) \tilde{\mathfrak{h}}_U(t) := \tilde{\mathfrak{h}}_U(t)$ and $\text{ad}^n(\mathfrak{h}_U(t)) \tilde{\mathfrak{h}}_U(t) := \text{ad}^1(\mathfrak{h}_U(t)) \text{ad}^{n-1}(\mathfrak{h}_U(t)) \tilde{\mathfrak{h}}_U(t)$. Alternatively, following Wei-Norman[28],

$$h(t) = \prod_i e^{-i\gamma_i(t) \mathfrak{g}_U^i} \quad (2.53)$$

where the $\gamma_i(t)$ are related to $\alpha_i(t)$ through a system of nonlinear differential equations. This form of $h(t)$ is particularly well-suited for parabolic decomposition and CS realizations.

The adjoint action on $\mathbf{F}_{\mathbb{S}}(G_A^{\mathbb{C}})$ induces a continuous, time-dependent unitary inner automorphism on $L_B(\mathcal{H})$ through the $*$ -representation

$$\begin{aligned} \mathcal{R}_{\lambda}^{(1)}(F) \mapsto \mathcal{R}_{\lambda}^{(1)}(F(t)) &:= \mathcal{R}_{\lambda}^{(1)}(\text{Ad}(h(t))F) = \rho(h(t)^{-1}) \left[\mathcal{R}_{\lambda}^{(1)}(F) \right] \rho(h(t)) \\ &= \text{Ad}(h(t)) \mathcal{R}_{\lambda}^{(1)}(F) \end{aligned} \quad (2.54)$$

where $\rho(h(t)) = e^{-i\rho'(\tilde{\mathfrak{h}}(t))} = \prod_i e^{-i\gamma_i(t) \rho'(\mathfrak{g}_i)}$.

The evolution operator is defined by $U(t) := \rho(h(t))$ which suggests to define a Schrödinger state-vector $\psi(t) := U(t)\psi$. Then transition amplitudes have a Heisenberg and Schrödinger representations as usual

$$\langle \phi | \mathcal{R}_{\lambda}^{(1)}(F(t)) \psi \rangle = \langle \phi | U(t)^{-1} \mathcal{R}_{\lambda}^{(1)}(F) U(t) | \psi \rangle = \langle \phi(t) | \mathcal{R}_{\lambda}^{(1)}(F) | \psi(t) \rangle. \quad (2.55)$$

Moreover the dynamics can be expressed in the CS model of a Schrödinger state-vector as expected

$$\frac{d\psi_{\mu}(t, z)}{dt} = -i\widehat{H(t)}\psi_{\mu}(t, z) \quad \forall t \in [t_a, t_b] \text{ and } z \in Z_{\partial} \quad (2.56)$$

where $H(t) = \rho'(\tilde{h}(t)) \in L(\mathcal{H})$. The CS model of H is a vector field on the group jet bundle restricted to Z .

The supposition that dynamics is governed by *inner* automorphisms has important consequences: Along with generating the dynamics of observables, $h(t)$ also induces an adjoint action on $U(\mathfrak{G}_A^{\mathbb{C}})$. The action represents evolution in $U(\mathfrak{G}_{A,\lambda}^{\mathbb{C}})$ because

$$\frac{d\rho'(\mathfrak{g}_U(t))}{dt} = i \left[\tilde{H}(t), \rho'(\mathfrak{g}_U(t)) \right]. \quad (2.57)$$

So in particular, $h(t)$ effects a change $p \mapsto \text{Ad}(h(t))p =: p(t)$ for all $p \in P$.

Recall that $\psi(zp) \in \mathcal{W}_z$ for all $p \in P$. But after evolution, $\psi(zp(t))$ is no longer necessarily an element of \mathcal{W}_z : Using (2.53) it is easy to see that there exist possible $h(t)$ such that $p(t) \notin P$ for some or all $t \in [t_a, t_b]$. If it happens that $\psi(zp(t)) \notin \mathcal{W}_z$, then it makes sense to define a new parabolic subgroup $\tilde{P}_{h(t)} = \text{Ad}(h(t))P$ along with its associated ground states. Consequently, *ground states depend on the evolution history of a closed system*. Moreover, $\tilde{P}_{h(t)}$ induces a new coset space $\tilde{Z}_{h(t)}$ with its associated CS model. In this sense Z evolves, and the physical interpretation of CS is time-dependent. In other words, the kinematics is time-dependent in general.

But what about the vacuum? By definition, the vacuum furnishes the trivial representation of $Sp(8, \mathbb{R})$ with $\langle \varphi_0 | \varphi_0 \rangle_{\mathcal{H}} = |\mathbf{w}_-|$. So, for the expectation of a unitary evolution (which is precipitated by an observation/measurement that induces the homomorphism $G_A^{\mathbb{C}} \rightarrow G_{A,\lambda}^{\mathbb{C}}$), the vacuum doesn't change.

However, we do not *a priori* exclude the possibility of non-unitary evolution of a closed quantum system that has been perturbed by an external agent. That is, we contemplate SQM of an open quantum system. The perturbation is still dictated by some subgroup $h(\mathbb{R}) \subset G_{A,\lambda}^{\mathbb{C}}$, but the associated representation is no longer necessarily unitary. We conjecture that in general this may lead to a new vacuum $\mathbf{v}_{\tilde{\mu}}$ induced from a new degenerate partition $\tilde{\mu} = [\tilde{\mu}, \tilde{\mu}, \tilde{\mu}, \tilde{\mu}]$. The vacuum module $\mathcal{W}_{(\tilde{\mu})}$ remains one-dimensional, but now $\langle \tilde{\varphi}_0 | \tilde{\varphi}_0 \rangle_{\mathcal{H}} = |\tilde{\mathbf{w}}_-| \neq |\mathbf{w}_-|$.

Remark 2.5 *This section was meant to outline the quantization of $Sp(8, \mathbb{R})$, but much of the construction was actually built using $Sp(8, \mathbb{C})$. This should not pose a problem as long as we are careful to restrict to real objects and/or subspaces at the appropriate times making use of the complex structure J .*

3 Some physical interpretations

Although there are still many aspects of the quantization that merit further investigation, we will move on to physical interpretation of SQM — the nature of which is

somewhat speculative.

3.1 ‘Expected geometry’

Consider the combination $[\mathfrak{e}_i, \mathfrak{e}_j^\dagger]$. Using (2.5) and the commutation relations it is straightforward to show that $[\mathfrak{e}_i, \mathfrak{e}_j^\dagger] = 4\delta_{ij}\mathfrak{h}_i$. From the construction of $\mathcal{V}_{(\mu)}^{(r)}$ and the definition of the ground state we know that $(\rho'(\mathfrak{h}_i)\psi_0)(z) = \lambda_i \mathbf{v}_{w_-}$ where $\mathbf{v}_{w_-} \in \mathcal{V}_{(\mu)}^{(r)}$ is a dominant-integral lowest-weight vector.²¹ For the symplectic group, eigenvalues λ_i of \mathfrak{h}_i are purely real or purely imaginary. Hence, the topology of the maximal abelian subgroup embedded in the group manifold is locally $\mathbb{T}^k \times \mathbb{R}^{k'}$ where $k + k' = 4$.²² In particular, for the CS vacuum \mathbf{v}_μ ,

$$(\rho'([\mathfrak{e}_i, \mathfrak{e}_j^\dagger])^2 \varphi_0)(z) = [\text{diag}(\underbrace{\lambda_1^2, \dots, \lambda_k^2}_k, \underbrace{-|\lambda_1^2|, \dots, -|\lambda_{k'}^2|}_{k'})]_{i,j} \mathbf{v}_\mu \quad (3.1)$$

where the eigenvalues λ_k ($\lambda_{k'}$) are real (respectively imaginary). The same reasoning applies to $[Ad(j)\mathfrak{e}_i, Ad(j)\mathfrak{e}_j^\dagger]$, and similar reasoning applies to $[\mathfrak{e}_{i,j}, \mathfrak{e}_{i,j}^\dagger]$.

These properties, together with the observation that e and j are the only two *inner* involutive automorphisms, motivate the definition of a pre-geometry.

Definition 3.1 *Let R_k with $k \in \{0, \dots, 4\}$ represent a region in $Sp(8, \mathbb{R})$ covered by a single coordinate chart, and choose an open region $U_i \subseteq R_k$. The pre-geometry $\mathcal{G}(k, k') \subset L_B(\mathcal{H})$ is generated by the image under ρ' of \mathfrak{e}_{ij} (identified with the associated ten left-invariant vector fields on U_i) together with their inner involutive automorphisms, i.e.*

$$\mathcal{G}(k, k') := \{E, E_i, E_{i,j}, E_{-i}, J\}, \quad k + k' = 4 \quad (3.2)$$

where $i \neq j \in \{1, 2, 3, 4\}$, $E := \rho(e) = Id$, $E_i = \rho'(\mathfrak{e}_i)$, $E_{i,j} = \rho'(\mathfrak{e}_{i,j})$, and $J = \rho(j)$.

Note that $E_{-i} = Ad(J)E_i = E_i^\dagger$. Also, $\mathcal{G}(k, k')$ when restricted to the vacuum subspace is an algebra because φ_0 is a $U(4)$ singlet.²³

When $k = 3$ and $k' = 1$, (with suitable normalization)

$$\frac{\langle \varphi_0 | \rho'(\{\mathfrak{e}_i, \mathfrak{e}_j^\dagger\}) \varphi_0 \rangle}{\langle \varphi_0 | \varphi_0 \rangle} = [\text{diag}(1, 1, 1, -1)]_{i,j} =: \langle \eta_{ij} \rangle. \quad (3.3)$$

²¹Recall that eigenvalues of \mathbf{v}_{w_-} with respect to \mathfrak{g}' are positive integers. But here we are dealing with ρ' , and there is no such restriction on its eigenvalues.

²²Unless there is some reason to exclude certain real/imaginary eigenvalue combinations, the group manifold of $Sp(8, \mathbb{R})$ comprises five domains locally characterized by the five different topologies of the abelian subgroup manifold. A thorough discussion of this and other aspects of evolution on non-compact group manifolds can be found in [7].

²³In this case, the algebra is closely related but strictly different from what is called geometric algebra in the literature. Specifically, $E_{i,j}$ is not the antisymmetric product of the E_i .

The definition is equally valid for ground states in each $\mathcal{V}_{(\mu)}^{(r)}$, and since $\mathcal{W}_{(\mu)}$ is a direct sum of $\mathcal{V}_{(\mu)}^{(r)}$ the definitions hold for the ground state $\mathbf{v}_{w_-} \in \mathcal{W}_{(\mu)}$ as well; although $\mathcal{G}(k, k')$ will no longer be an algebra with respect to the ground state subspace.

For a non-trivial evolution where $Ad(h(t))\mathbf{e}_i = \mathbf{e}_i(t) \notin \mathfrak{Z}_-$,

$$\langle \eta_{ij}^{(h(t))} \rangle := \frac{\langle \psi_0 | U^{-1}(t) \rho'(\{\mathbf{e}_i, \mathbf{e}_j^\dagger\}) U(t) \psi_0 \rangle}{\langle \psi_0 | \psi_0 \rangle} = \frac{\langle \psi_{h(t)} | \{E_i, E_{-j}\} \psi_{h(t)} \rangle}{\langle \psi_0 | \psi_0 \rangle} \neq \langle \eta_{ij} \rangle. \quad (3.4)$$

Similarly, non-trivial dynamics induces a non-trivial almost complex structure

$$\langle J^{(h(t))} \rangle := \frac{\langle \psi_{h(t)} | J \psi_{h(t)} \rangle}{\langle \psi_0 | \psi_0 \rangle}, \quad (3.5)$$

and a non-trivial antisymmetric form

$$\langle \Omega_{ij}^{(h(t))} \rangle := \frac{\langle \psi_{h(t)} | (E_{i,j} - E_{j,i}) \psi_{h(t)} \rangle}{\langle \psi_0 | \psi_0 \rangle}. \quad (3.6)$$

The VEVs of the pre-geometry $\mathcal{G}(k, k')$ characterize the geometry of a complex manifold $\langle \rho(Ad(g)z) \rangle =: \langle Z^{(g)} \rangle$ (assuming it is a topological space), that can be interpreted as a phase-space state in the sense of the GNS construction. For want of a better name, we will call it the ‘expected phase space’.²⁴ To see this, notice that $\langle E_{ij}^{(g)} \rangle$ can only depend parametrically on \mathbf{Z} because $\rho'(\mathbf{p})\psi_0$ only transforms the $|\mu\rangle$ component of $\psi_0(z)$. Indeed, $\rho'(\mathbf{e}_{ij}^\dagger)$ annihilates the ground state, $\rho'(\mathbf{i}_{ij})$ is unitary, and $\rho'([\mathbf{e}_i, \mathbf{e}_j^\dagger])$ is normal. Hence, $Ad(p)\rho'([\mathbf{e}_i, \mathbf{e}_j^\dagger])$ can be diagonalized by a unitary similarity transformation on $\mathcal{W}_{(\mu)}$, which implies that a gauge transformation just corresponds to a change of coordinate basis in $\mathcal{W}_{(\mu)}$.

Accordingly, expected phase space has $\dim_{\mathbb{R}}(Sp(8, \mathbb{R})/U(4)) = 20$ and the associated CS model of the operators $Z^{(h(t))}$ encode the time-dependence of the geometry through their spectra.²⁵ Moreover, since $\psi_{gp} \sim \psi_g$ for all $p \in P$, the geometry is obviously gauge invariant. In other words, in a CS model non-trivial expected geometry is parametrized by \mathbf{Z} .

For a system near the ground state, one might expect the influence of the operators $E_{i,j}$ on the phase-space geometry to be very small if the $\mathbf{e}_{i,j}$ excitations require substantially higher energy. In this case it would make sense to integrate the pre-geometric structures over the off-diagonal variables to obtain an effective description.

Specifically, assume $\langle Z \rangle$ is a topological space and let $\mathbb{K} := \langle K \rangle \subset \langle Z \rangle$ denote the 4-d ‘diagonal’ subspace in $\langle Z \rangle$. Let $\langle \eta_{ij}^{(k)} \rangle$ denote $\langle \eta_{ij}^{(z)} \rangle$ integrated over off-diagonal variables in \mathbf{Z} ; likewise for $\langle J^{(k)} \rangle$ and $\langle \Omega_{ij}^{(k)} \rangle$. The symmetric form $\langle \eta_{ij}^{(k)} \rangle =: g_{ij}(k)$

²⁴In the limit of large systems, ‘expected phase space’ becomes classical, but a phase-space state would certainly not look classical for all systems. Note that η and Ω are not related through J so the expected geometry is not Kähler in general.

²⁵An open question is the physical meaning of the pre-geometric algebra $\mathcal{G}(k, k')$ when $k \in \{0, 2, 4\}$.

functions as a metric on the 4-d truncated space \mathbb{K} . Similarly, $\langle J^{(k)} \rangle =: J(k)$ and $\langle \Omega_{ij}^{(k)} \rangle =: \Omega_{ij}(k)$ are almost complex and symplectic structures on \mathbb{K} .

Evidently, in this effective description, $\langle \psi_{\sim 0} | \{E_i, E_{-j}\} \psi_{\sim 0} \rangle$ along with $\mathcal{G}(3, 1)$ could be used for a model of an expected 4-d space-time with approximate Poincaré symmetry. In this sense, 4-d space-time owes its existence to approximation; as does Poincaré symmetry and its irreducible representations.

On the other hand, expectations of the $E_{i,j}$ can be interpreted as directed-area elements according to the structure of the pre-geometry: So it may be that physics based on 4-d space-time is actually a truncation of a more accurate 10-d model — one in which both $\eta_{ij}^{(z)}$ and $\Omega_{ij}^{(z)}$ participate independently. For meso/macroscopic systems, it is not hard to imagine that $E_{i,j}$ operators can have significant expectation, and one can see the seeds of vortex-type dynamics that are algebraically independent from linear-type dynamics as long as $\langle (E_{i,j} - E_{j,i}) \rangle \neq \langle E_i \rangle \wedge \langle E_{-j} \rangle$.

3.2 Particles and fields

The fact that P generates $\mathcal{V}_{(\mu)}^{(r)}$ and the requirement that physical states are covariant under right-translation by P motivate the interpretation of ‘particles’ and ‘fields’:

Definition 3.2 *An elementary/quasi particle is an eigenvector $\mathbf{v}_p \in \mathcal{V}_{(\mu)}^{(r)}$ of $\rho'(\mathfrak{h}_i)$ for each $i \in \{1, 2, 3, 4\}$. In the CS model, an elementary field at a point $z \in Z^{\mathbb{C}}$ is defined by*

$$\psi_{\mathbf{v}_p}(z) := (z; \mathbf{v}_p | \psi) = \int_{Z^{\mathbb{C}} \setminus S^n} (z; \mathbf{v}_p | z'^*; \boldsymbol{\mu}) d\boldsymbol{\sigma}(z') \psi_{\boldsymbol{\mu}}(z') \quad (3.7)$$

and satisfies $\widehat{\rho'(\mathfrak{h}_i)} \psi_{\mathbf{v}_p}(z) = \lambda_i \psi_{\mathbf{v}_p}(z)$. We call $|\lambda_i|$ (suitably normalized) the $U(4)$ charges.

Assuming non-degenerate eigenvectors, this can be inverted

$$\psi_{\boldsymbol{\mu}}(z) = \int_{Z^{\mathbb{C}} \setminus S^n} (z; \boldsymbol{\mu} | z'^*; \mathbf{v}_p) d\boldsymbol{\sigma}(z') \psi_{\mathbf{v}_p}(z') \quad (3.8)$$

and $\psi_{\boldsymbol{\mu}}(z)$ interpreted as a superposition of elementary fields. Thus the CS model of a state-vector is a field; albeit not elementary.²⁶

This is an obvious definition based on (2.8). That is, elementary particles span the full module \mathcal{V} of relevant representations as required. But — contrary to the weight decomposition of \mathcal{V} which leads to the irreducible discrete series representations and,

²⁶We should emphasize this definition of field does not coincide with the usual definition in QFT. Our fields are a generalization of the wave function in ordinary QM. In QFT, a field is a superposition of creation and/or annihilation operators $c_{\alpha}, c_{\alpha}^{\dagger}$, and in SQM there is no object to directly compare since everything is constructed from products $c_{\alpha,\beta}$. To compare indirectly, the commutator of QFT fields roughly corresponds to the CS model of the corresponding SQM operator. In the other direction, our fields are a CS model of a multi-particle state $(c_{\alpha_1} c_{\alpha_2} \cdots c_{\alpha_{n-1}}^{\dagger} c_{\alpha_n}^{\dagger}) \varphi_o =: \psi$.

hence, particles characterized solely by their $U(4)$ partition — the dominant-integral lowest-weight parabolic decomposition characterizes particles by their $U(4)$ charges *and* their P representation. And, according to remark 2.3, the conjugate CS model represents anti-particles.

Notice the definition holds for all relevant representations labeled by r . Since $\rho'(\mathfrak{h}_i)$ can be interpreted as a number operator, multi-particle states are accounted for by appropriate $\mathcal{V}_{(\mu)}^{(r)}$ associated with tensor products and direct sums. For example, by definition the vacuum is an elementary particle that happens to be the degenerate $U(4)$ representation labeled by some partition $\mu = [\mu, \mu, \mu, \mu]$. Since there is no *a priori* reason to settle on a special value for μ , It is not hard to imagine that different closed dynamical systems could have different vacua. After the vacuum, the next simplest irrep is the defining rep of $U(4)$. It is standard to interpret anti-symmetric tensor products of this irrep with matter particles/fields. Moving on to the adjoint representation, the CS model of state-vectors in the adjoint representation can be interpreted as gauge fields. Elementary gauge bosons are then naturally identified with eigenvectors of \mathfrak{h}_i in the adjoint representation. As the rank of $Sp(8, \mathbb{R})$ is four, there are 26 elementary gauge bosons that are characterized by four types of charge. Consequently, according to (2.18), $\{\pi'_z(\rho'(\mathfrak{P}))\}$ are gauge potentials on Z .²⁷ It is remarkable that these gauge potentials possess both external and internal structure, and in a CS realization each component is a matrix-valued differential operator whose dimension is dictated by the $\mathcal{W}_{(\mu)}$. Presumably, symmetric tensor products of the adjoint representation yield multi-boson particles/fields. Finally, the last two basic representations $\mathcal{V}_{48}^{(3)}$ and $\mathcal{V}_{42}^{(4)}$ are suspected to be relevant, but their physical meaning is unclear.

3.3 CS phase space

The definition of field was given in terms of CS parametrized by 10 complex coordinates. We want to now express them in terms of 20 real parameters. For this purpose, use a matrix phase space CS model. This effectively separates the ‘internal’ gauge freedom from the ‘external’ gauge freedom and more closely corresponds to observed fields in terrestrial particle physics.

To construct the model, make use of remark 2.2 to construct a phase space vector bundle \mathcal{I} and its associated CS model by inducing a representation from $U(4)$. Define the operators $Q_{ij} := \rho'(\mathfrak{e}_{ij} + \mathfrak{e}_{ij}^\dagger)/2$ and $\Pi_{ij} := \rho'(\mathfrak{e}_{ij} - \mathfrak{e}_{ij}^\dagger)/2$. Then a phase space CS

²⁷These are the closest analog to fields in QFT. Presumably, one could use these operators along with $\{\pi'_z(\rho'(\mathfrak{Z}_+))\}$ to make contact with field theory.

can be defined by

$$\begin{aligned}
|q, \pi; \boldsymbol{\mu}\rangle &:= \left(\exp \left\{ \frac{1}{4} \sum_{i,j} (z_{ij} \mathbf{e}_{ij} + z_{ij}^* \mathbf{e}_{ij}^\dagger) \right\} \right) |\boldsymbol{\mu}\rangle \\
&= \left(\exp \left\{ \frac{1}{2} \sum_{i,j} q_{ij} (\mathbf{e}_{ij} + \mathbf{e}_{ij}^\dagger) + i\pi_{ij} (\mathbf{e}_{ij} - \mathbf{e}_{ij}^\dagger) \right\} \right) |\boldsymbol{\mu}\rangle
\end{aligned} \tag{3.9}$$

with $z_{ij} = q_{ij} + i\pi_{ij}$. CS realizations of other phase space objects can be defined along the same lines as before. However, cross-sections of \mathcal{I} cannot be directly identified with state-vectors. For that we need to single out a distinguished polarization on $Sp(8, \mathbb{C})/U(4)$ that allows determination of a Lagrangian subspace.²⁸ Then the coordinates on the subspace can be consistently identified with the spectrum of ten mutually-commuting operators in \mathfrak{z} . Consequently, given a polarization, the CS matrix picture of a state-vector is $\boldsymbol{\psi}_\mu(\mathbf{Q}, \boldsymbol{\Pi})$, and the notion of elementary particles and fields applies also here.

As discussed previously, we want to associate generators of ‘external’ dynamics with

$$Q + \Pi := \begin{pmatrix} Q_1 & Q_{12} & Q_{13} & Q_{14} \\ Q_{12} & Q_2 & Q_{23} & Q_{24} \\ Q_{13} & Q_{23} & Q_3 & Q_{34} \\ Q_{14} & Q_{24} & Q_{34} & Q_4 \end{pmatrix} + \begin{pmatrix} \Pi_1 & \Pi_{12} & \Pi_{13} & \Pi_{14} \\ \Pi_{12} & \Pi_2 & \Pi_{23} & \Pi_{24} \\ \Pi_{13} & \Pi_{23} & \Pi_3 & \Pi_{34} \\ \Pi_{14} & \Pi_{24} & \Pi_{34} & \Pi_4 \end{pmatrix}, \tag{3.10}$$

and generators of ‘internal’ dynamics with

$$M + A := \begin{pmatrix} M_1 & M_{12} & M_{13} & M_{14} \\ M_{12} & M_2 & M_{23} & M_{24} \\ M_{13} & M_{23} & M_3 & M_{34} \\ M_{14} & M_{24} & M_{34} & M_4 \end{pmatrix} + \begin{pmatrix} A_1 & A_{12} & A_{13} & A_{14} \\ A_{21} & A_2 & A_{23} & A_{24} \\ A_{31} & A_{32} & A_3 & A_{34} \\ A_{41} & A_{42} & A_{43} & A_4 \end{pmatrix} \tag{3.11}$$

where

$$\begin{aligned}
Q_i &= \rho'(\mathbf{e}_i + \mathbf{e}_i^\dagger)/2 & \Pi_i &= \rho'(\mathbf{e}_i - \mathbf{e}_i^\dagger)/2 \\
Q_{i,j} &= \rho'(\mathbf{e}_{i,j} + \mathbf{e}_{i,j}^\dagger)/2 & \Pi_{i,j} &= \rho'(\mathbf{e}_{i,j} - \mathbf{e}_{i,j}^\dagger)/2
\end{aligned} \tag{3.12}$$

while the operators representing $\mathfrak{u}(4)$ are

$$\begin{aligned}
M_i &= \rho'(\mathfrak{h}_i + \mathfrak{h}_i^\dagger)/2 & A_i &= \rho'(\mathfrak{h}_i - \mathfrak{h}_i^\dagger)/2 = 0 \\
M_{i,j} &= \rho'(e_{i,-j} + e_{i,-j}^\dagger)/2 & A_{i,j} &= \rho'(e_{i,-j} - e_{i,-j}^\dagger)/2.
\end{aligned} \tag{3.13}$$

²⁸Explicitly, choose a connection on $Sp(8, \mathbb{R})$. Use it to map the completely integrable distribution associated with $T(Sp(8/\mathbb{R})/U(4))$ to a completely integrable horizontal distribution $H \subset T(Sp(8, \mathbb{R}))$. Identify the horizontal subspace with a suitable linear combination of ten mutually commuting generators. Conclude that a connection is required to fix a polarization.

Note that

$$\begin{aligned} Q^\dagger &= Q & \Pi^\dagger &= -\Pi \\ M^\dagger &= M & A^\dagger &= -A . \end{aligned} \quad (3.14)$$

As expected, a gauge picture emerges with a choice of Lagrangian subspace on the base manifold $(\mathbf{Q}, \mathbf{\Pi})$. With the canonical choice, the CS models of $\{\Pi, M, A\}$ become gauge potentials while the q_{ij} parametrize the configuration space. Interpret $\underline{\Pi} := \Pi + A$ as ‘interaction momentum’ and $\underline{Q} := Q + M$ as ‘interaction configuration’ operators. This motivates the definition

Definition 3.3 *The stress-energy operator $T \in L(\mathcal{H})$ is defined by*

$$\underline{T} := \frac{1}{2} (\{\underline{\Pi}, \underline{\Pi}^\dagger\} + \{\underline{Q}, \underline{Q}^\dagger\}) . \quad (3.15)$$

In the CS matrix picture, \underline{T} is a $4N \times 4N$ second-order partial differential operator where $N := \dim_{\mathbb{R}}(\mathcal{W}_{(\mu)})$. This suggests the interpretation of the *classical* stress-energy *matrix* for a non-trivial system;

Definition 3.4 *The classical stress-energy²⁹ is defined to be*

$$\underline{T}(g) := \langle \underline{T}^{(g)} \rangle = \langle \psi_0 | U(g)^* \underline{T} U(g) | \psi_0 \rangle = \langle \psi_g | \underline{T} | \psi_g \rangle \quad (3.16)$$

where ψ_0 is the ground state associated with $\mathbf{v}_{w_-} \in \mathcal{W}_{(\mu)}$.

For stress-energy CS eigenfunctions $\psi_{\mu}^{(\kappa)}(q, \pi)$, this gives an interpretation of energy;

$$\text{tr} \left(\widehat{\underline{T}^{(g)}} \right) \psi_{\mu}^{(\kappa)}(q, \pi) =: E_{\kappa} \psi_{\mu}^{(\kappa)}(q, \pi) . \quad (3.17)$$

A more informative object is the CS eigenfunction in the matrix picture $\psi_{\mu}^{(\kappa)}(\mathbf{Q}, \mathbf{\Pi})$ defined by

$$\widehat{\underline{T}^{(g)}} \psi_{\mu}^{(\kappa)}(\mathbf{Q}, \mathbf{\Pi}) = \kappa^2 \psi_{\mu}^{(\kappa)}(\mathbf{Q}, \mathbf{\Pi}) \quad (3.18)$$

where κ^2 is a 4×4 matrix with real entries and $E_{\kappa} = \text{tr}(\kappa^2)$. This motivates the identification $D_t \equiv d/dt + i \text{ad}(\widetilde{\underline{Q}} + i \widetilde{\underline{\Pi}})$ for the derivative operator in prop. 3.1 in the context of CS phase space.

Of course, evolution will generally alter the form of $\widehat{\underline{T}^{(g)}}$ rendering $\psi_{\mu}^{(\kappa)}(\mathbf{Q}, \mathbf{\Pi})$ no longer stationary. But then one can define a new parabolic decomposition and retrace the quantization procedure to arrive at a new description of a quadratic Hamiltonian

²⁹An interesting aspect that we won’t address here is how to calculate $\langle \underline{T}^{(g)} \rangle$. However, taking a sum-over-paths approach via functional integration would seem to indicate a contribution from each of the five domains in the group manifold possessing the topology $\mathbb{T}^k \times \mathbb{R}^{k'}$. For the vacuum, equal contributions from each domain would appear to cancel. This issue clearly merits further investigation.

operator and its CS model of eigenstates. Needless to say, physical interpretation relative to the new parabolic decomposition may be highly nontrivial.

Like the symmetric, complex, and symplectic forms; the classical stress-energy is actually a function of just q due to gauge covariance of the ground state. The form of \mathfrak{Z}_+ leads naturally to the interpretation that $\underline{T}^{(g)}$ referred to a CS induced truncated cotangent bundle *near the ground state* corresponds to a stress-energy space-time tensor $\underline{T}(q)$ when $k \in \{1, 3\}$. This is closely related to the symmetric, almost complex, and symplectic structures defined on \mathbb{K} in subsection 3.1, and it immediately suggests the system Hamiltonian operator be identified with $\text{tr}(\underline{T}(q))$ where tr denotes the trace over i, j indices when it exists.

Remark 3.1 *Although this subsection dealt with phase space associated with real polarizations, similar considerations could be applied to holomorphic polarizations. Presumably, this would lead to a Segal-Bargmann holomorphic phase space construction.*

3.4 Matrix quantum mechanics

Conspicuously absent from the elementary particle tally are the generators of \mathfrak{Z}_+ . This of course is due to the fact that their role is to parametrize CS via \mathbf{Z} . Nevertheless, for time-independent \tilde{H} , (2.57) implies the \mathfrak{Z}_+ generators evolve according to the second-order operator equation

$$\frac{d^2 E_{ij}^\dagger(t)}{dt^2} + ad^2(\tilde{H})E_{ij}^\dagger(t) = 0. \quad (3.19)$$

where $E_{ij}^\dagger := \rho'(\mathfrak{e}_{ij}^\dagger) \in L(\mathcal{H})$ are a set of ten operators.

Since the $E_{ij}^\dagger(t)$ mutually commute, consider the eigenstates $E_{ij}^\dagger(t) \Psi^{(\lambda)} = \lambda_{ij} \Psi^{(\lambda)}$. In the CS model, this becomes

$$\widehat{E_{ij}^\dagger(t)} \Psi_\mu^{(\lambda)}(z) = \lambda_{ij} \Psi_\mu^{(\lambda)}(z) \quad (3.20)$$

where the CS models of the operators are now $N \times N$ matrices with $N = \dim_{\mathbb{C}}(\mathcal{W}_{(\mu)})$. So, (3.19) and its first-order equivalent (2.57) — referred to an eigenstate basis in \mathcal{H} — looks like matrix quantum mechanics. This type of matrix equation is notoriously difficult to handle and yet simple enough that general qualitative information can be gleaned by inspection. Clearly N can grow very large for multi-particle states, and for macro systems taking $N \rightarrow \infty$ is a reasonable approximation. So off hand, it appears that the \mathfrak{Z}_+ sector of SQM will look like a quantum membrane theory in 10-d for multi-particle systems.

For first-order $\tilde{\mathfrak{h}} = \sum_{k,l} \beta_{kl} \mathfrak{g}^{kl}$ with $\tilde{\mathfrak{h}}^\dagger = \tilde{\mathfrak{h}}$, (3.19) can be written as a Lagrangian density (assuming time-independent $\tilde{\mathfrak{h}}$)

$$\begin{aligned} \mathcal{L}_\epsilon(t) &= \frac{1}{2} \text{Tr} \left\{ B \left(\dot{\mathfrak{e}}_{ij}(t), \dot{\mathfrak{e}}_{kl}^\dagger(t) \right) + B \left(\text{ad}(i\tilde{\mathfrak{h}})\mathfrak{e}_{ij}(t), \text{ad}(i\tilde{\mathfrak{h}})^\dagger \mathfrak{e}_{jk}^\dagger(t) \right) \right\} \\ &=: \frac{1}{2} \text{Tr} \left\{ B \left(D_t \mathfrak{e}_{ij}, D_t^\dagger \mathfrak{e}_{kl}^\dagger \right) \right\} \end{aligned} \quad (3.21)$$

where B is the Cartan-Killing form on the Lie algebra, $D_t := d/dt + \text{ad}(i\tilde{\mathfrak{h}})$, and $\tilde{\mathfrak{h}}^\dagger = \tilde{\mathfrak{h}}$ was used in the third line. In particular, suppose $\tilde{\mathfrak{h}} \in \mathfrak{P}$ is pure ‘internal’ gauge, i.e. $\tilde{\mathfrak{h}} \in \mathfrak{U}(4)$. Then $\tilde{\mathfrak{h}} \sim \sum_{k,l} \tilde{\mathfrak{h}}_{kl}$ with $\tilde{\mathfrak{h}}^\dagger = \tilde{\mathfrak{h}}$ as required, and it follows from $[\mathfrak{h}_{ij}, \mathfrak{e}_{kl}^\dagger] \in \mathfrak{Z}_+$ that $[d\mathfrak{e}_{ij}^\dagger/dt, \mathfrak{e}_{ij}^\dagger] = 0$. At the other extreme, if the Hamiltonian is ‘external’ in the sense that $\tilde{\mathfrak{h}} \sim \sum_{k,l} (\tilde{\mathfrak{e}}_{kl} \pm \tilde{\mathfrak{e}}_{kl}^\dagger)$, then $[d\mathfrak{e}_{ij}^\dagger/dt, \mathfrak{e}_{ij}^\dagger, \mathfrak{e}_{ij}^\dagger] = 0$.

The Lagrangian density yields the evolution equation for \mathfrak{e}_{ij} as well. Obviously, the CS model of the evolution equation of E_{ij} referred to the eigenfunctions $\Psi^{(\lambda)}$ is more complicated than simple matrix quantum mechanics. However, being mutually commuting, E_{ij} possess a different eigenbasis where their evolution is governed by matrix quantum mechanics. The physical interpretation of such eigenstates is not clear: nevertheless, they may provide a possibly interesting dual picture.

The same Lagrangian density can be used for the entire Lie algebra.

Proposition 3.1 *Let $\mathfrak{g}_{ij} \in \mathfrak{Sp}(8)$ and suppose $\tilde{\mathfrak{h}}$ is a time-independent, first-order evolution generator. The Lagrangian density that generates evolution in $\mathfrak{Sp}(8)$ induced by $\tilde{\mathfrak{h}}$ is given by*

$$\mathcal{L}_{\mathfrak{g}}(t) = \frac{1}{2} \text{tr} \left\{ B \left(D_t \mathfrak{g}_{ij}(t), D_t^\dagger \mathfrak{g}_{kl}^\dagger(t) \right) \right\} \quad (3.22)$$

where B the Cartan-Killing metric on $\mathfrak{Sp}(8)$ and $D_t := d/dt + \text{ad}(i\tilde{\mathfrak{h}})$.

Alternatively, it can be promoted to the Hilbert space and expressed in terms of operators $G_{ij} := \rho'(\mathfrak{g}_{ij}) \in L(\mathcal{H})$ as

$$\mathcal{L}_G(t) = \frac{1}{2} \langle \psi_0 | |D_t G_{ij}(t)|^2 | \psi_0 \rangle. \quad (3.23)$$

This suggests an analogous Lagrangian density formulation for the Heisenberg equation for bounded operators $O_\lambda^{-1} = \mathcal{R}_\lambda^{-1}(O)$ associated with observables $O \in \mathbf{F}_\mathbb{S}(G_A^\mathbb{C})$.

$$\mathcal{L}_{O_\lambda^{-1}}(t) = \frac{1}{2} \langle \psi_0 | |D_t O_\lambda^{-1}(t)|^2 | \psi_0 \rangle. \quad (3.24)$$

Given that $Sp(8, \mathbb{C})$ can be identified with the cotangent bundle T^*Q for suitable $Q \subset Sp(8, \mathbb{R})$ and the fact that $\rho(G_A^\mathbb{C}) \subset U(\mathcal{R}_\lambda^{(1)}(\mathbf{F}_\mathbb{S}(G_A^\mathbb{C})))$ coming from prop. 2.3, the form of these Lagrangians begs to formulate the dynamics of SQM as Hamiltonian mechanics of a $U(4)$ gauge theory on a non-commutative phase space $T^*\rho(Q)$. Unfortunately, a proper treatment of this notion lies outside our present scope.

3.5 Classical SQM

Consider a closed and bounded quantum system in the CS phase space model. Write $\underline{T}^{(g)} = \underline{T} + V(g)$ with $V(g) := \underline{T}^{(g)} - \underline{T}$. Assume there exist CS eigenfunctions $\underline{\hat{T}} \psi_\mu^{(\kappa)}(q, \pi) = \kappa^2 \psi_\mu^{(\kappa)}(q, \pi)$ that realize a complete set of eigenstates in \mathcal{H}_Q with a

discrete spectrum (since the system is bounded). Because $\text{tr}(\underline{T}) \sim H^2$, these eigenfunctions are associated with elementary particles according to the definition.³⁰ Then, for a Schrödinger CS state-vector, we have

$$\psi_\mu(t, q, \pi) := \sum_{\kappa} c_\kappa(t) \psi_\mu^{(\kappa)}(q, \pi) := (q, \pi; \boldsymbol{\mu} | \sum_{\kappa} c_\kappa(t) \psi_\mu^{(\kappa)}) . \quad (3.25)$$

Use the CS resolution of the identity to write

$$c_\kappa(t) = \langle \psi_\mu^{(\kappa)} | \psi(t) \rangle = \int_{U_i} \psi_\mu^{(\kappa)\dagger}(q, \pi) \mathbf{d}\sigma(q, \pi) \psi_\mu(t, q, \pi) . \quad (3.26)$$

In particular,

$$\begin{aligned} N_{\kappa, \kappa'} = \langle \psi_\mu^{(\kappa)} | \psi_\mu^{(\kappa')} \rangle &= \int_{U_i} \psi_\mu^{(\kappa)\dagger}(q, \pi) \mathbf{d}\sigma(q, \pi) \psi_\mu^{(\kappa')}(q, \pi) \\ &= \int_{U_i} \psi_\mu^{(\kappa)\dagger}(q, \pi) \mathbb{P}(q, \pi) \psi_\mu^{(\kappa')}(q, \pi) d(q, \pi) \end{aligned} \quad (3.27)$$

can be interpreted as the number density function in the spectrum space of \underline{T} . Consequently, under suitable conditions $\psi_\mu^{(\kappa)}(q, \pi)$ can be interpreted as a CS distribution with an associated probability measure $\mu(q, \pi) := \psi_\mu^{(\kappa)\dagger}(q, \pi) \sigma(q, \pi) \psi_\mu^{(\kappa')}(q, \pi) / N_{\kappa, \kappa'}$ which means that $\text{tr}(\psi_\mu^{(\kappa)\dagger}(q, \pi) \mathbb{P}(q, \pi) \psi_\mu^{(\kappa')}(q, \pi))$ is the number density of particle-type κ on phase space $(\mathbf{Q}, \boldsymbol{\Pi})$.

Evidently $\psi_\mu(t, q, \pi)$ is a time-dependent superposition of elementary particle state-vectors so it permits the notion of particle creation and annihilation. To see this in more detail, it is convenient to go to the interaction picture defined by

$$\Psi_\mu(t, q, \pi) := e^{-it\hat{\underline{T}}} \psi_\mu(t, q, \pi) \quad (3.28)$$

and

$$V_I^{(g)}(t) := e^{-it\mathbf{T}} V^{(g)} e^{it\mathbf{T}} . \quad (3.29)$$

Then

$$\partial_t c_\kappa(t) = \sum_{\kappa'} (V_I^{(g)}(t))_{\kappa, \kappa'} c_{\kappa'}(t) \quad (3.30)$$

and

$$\frac{dO_I(t)}{dt} = i[\underline{T}, O_I(t)] \quad (3.31)$$

with $(V_I^{(g)}(t))_{\kappa, \kappa'} := \langle \psi_\mu^{(\kappa)} | V_I^{(g)}(t) | \psi_\mu^{(\kappa')} \rangle$ together describe the dynamics.

There are two observations to make. First, recall the definition of $\mathbb{P}(q, \pi)$: It is related to the CS overlap kernel so it will be time-dependent in general. This follows

³⁰In fact, in hind sight it might be better to use this as the definition of elementary particles.

because the same adjoint group action that induces dynamics will *potentially* induce a change in the overlap kernel. Consequently, the particle content of $\psi_\mu(t, q, \pi)$ changes because the coefficients $c_\kappa(t)$ are time-dependent *and* the particle density $\text{tr}(\psi_\mu^{(\kappa)\dagger}(q, \pi) \mathbb{P}(q, \pi) \psi_\mu^{(\kappa)}(q, \pi))$ can also change. A time-dependent particle density on phase space is in stark contrast to elementary quantum mechanics, and it owes its existence to the $U(4)$ subgroup of dynamical $Sp(8, \mathbb{R})$ — indeed, $U(4)$ singlet states have constant particle density since the overlap kernel is trivial in this case.

The second observation concerns the CS model of the interaction Heisenberg equation. In the eigenfunction basis, the operator is realized as a matrix that will become very large for many-particle systems. It is known that the commutator approaches the Poisson bracket in this limit. This brings us to the correspondence principle.

Passage from the quantum to a classical phase space description via the correspondence principle is standard, but it yields a non-standard result. To simplify notation slightly, restrict to the case of $U(4)$ singlets;

Proposition 3.2 *Suppose the dynamics of a closed and bounded system is determined by $\underline{T}^{(g)} = H(Q, \Pi)$ with H a self-adjoint operator that is bounded from below. The ‘classical dynamics’ relative to a CS of particle-type κ is expressed by matrix Hamilton’s equations*

$$\begin{aligned} \frac{d\pi^{(\kappa)}(t)}{dt} &= \{\pi(t), H(\mathbf{q}(t), \pi(t))\}^{(\kappa)} \\ \frac{d\mathbf{q}^{(\kappa)}(t)}{dt} &= \{\mathbf{q}(t), H(\mathbf{q}(t), \pi(t))\}^{(\kappa)} \end{aligned} \quad (3.32)$$

with $\pi^{(\kappa)}(t) := \langle \psi_\mu^{(\kappa)} | \Pi(t) \psi_\mu^{(\kappa)} \rangle / N_{\kappa, \kappa}$ and $\mathbf{q}^{(\kappa)}(t) := \langle \psi_\mu^{(\kappa)} | Q(t) \psi_\mu^{(\kappa)} \rangle / N_{\kappa, \kappa}$, and the ‘classical’ bracket³¹ relative to the κ -type CS is defined by

$$\begin{aligned} \{\mathbf{a}, \mathbf{b}\}^{(\kappa)} &:= \langle \psi_\mu^{(\kappa)} | [A, iB] \psi_\mu^{(\kappa)} \rangle / N_{\kappa, \kappa} \\ &= \frac{1}{N_{\kappa, \kappa}} \int_{Z_\theta} \psi_\mu^{(\kappa)\dagger}(q, \pi) \mathbb{P}(q, \pi) [\widehat{A}, i\widehat{B}] \psi_\mu^{(\kappa)}(q, \pi) d(q, \pi). \end{aligned} \quad (3.33)$$

³¹We stress that the classical bracket only approaches the Poisson bracket as the system becomes macroscopic. In particular, the classical bracket can be applied to mesoscopic systems characterized by elementary/quasi particles of κ -type.

Proof: This is just Ehrenfest's theorem:

$$\begin{aligned}
\frac{d\boldsymbol{\pi}^{(\kappa)}}{dt} &:= \left\langle \boldsymbol{\psi}_{\mu}^{(\kappa)} \left| \frac{d\Pi}{dt} \boldsymbol{\psi}_{\mu}^{(\kappa)} \right. \right\rangle / N_{\kappa, \kappa} \\
&= \left\langle \boldsymbol{\psi}_{\mu}^{(\kappa)} \left| [\Pi, iH(Q, \Pi)] \boldsymbol{\psi}_{\mu}^{(\kappa)} \right. \right\rangle / N_{\kappa, \kappa} \\
&= \frac{1}{N_{\kappa, \kappa}} \int_{Z_{\partial}} \boldsymbol{\psi}_{\mu}^{(\kappa)\dagger}(q, \pi) \mathbb{P}(q, \pi) \left[\widehat{\Pi}, i\widehat{H(Q, \Pi)} \right] \boldsymbol{\psi}_{\mu}^{(\kappa)}(q, \pi) d(q, \pi) \\
&=: \{ \boldsymbol{\pi}, H(\boldsymbol{q}, \boldsymbol{\pi}) \}^{(\kappa)}
\end{aligned} \tag{3.34}$$

and we interpret the matrix $H(\boldsymbol{q}, \boldsymbol{\pi})$ as the classical Hamiltonian.

Recall that the induced representation was built from $L^2(Z)$ equivariant p-forms so the CS are properly normalized and the integral well-defined (assuming the eigenfunctions $\boldsymbol{\psi}_{\mu}^{(\kappa)}$ are indeed a complete set). \square

Evidently the classical object $\boldsymbol{\pi}^{(\kappa)}(t)$ is a symmetric 4×4 matrix of momenta, and the classical equations reduce to Hamilton's equations under special conditions. For example, consider a system with ground state ψ_0 and suppose $k = \{1, 3\}$. Assume the evolution is 'mild' in the sense that $\boldsymbol{\psi}_{\mu}^{(\kappa)}(q, \pi)$ is annihilated by the off-diagonal components of Q and Π . Then (3.34) reduces to standard Hamilton's equations

$$\begin{aligned}
\frac{dp_i^{(\kappa)}(t)}{dt} &= \{p_i(t), H(q, p)\}^{(\kappa)} =: F_i^{(\kappa)}(q, p; t), \quad i \in \{0, 1, 2, 3\} \\
\frac{dq_i^{(\kappa)}(t)}{dt} &= \{q_i(t), H(q, p)\}^{(\kappa)}.
\end{aligned} \tag{3.35}$$

Now define the system configuration space \mathcal{Q} to be the spectrum $\mathcal{Q} := \sigma(Q)$. Let \mathcal{P} denote the spectrum $\sigma(\Pi)$ and construct the associated classical phase space³² $\mathcal{Q} \times \mathcal{P}$. A trivial remark that should be stressed is that a generic classical phase space will not be a smooth manifold. If it is smooth, equip \mathcal{Q} with the metric $\langle \eta_{ij} \rangle$ and identify evolution-time with proper time η_{00} . Then (3.35) can be interpreted as Hamilton's equations on the classical 4-d space-time cotangent bundle $T^*\mathcal{Q} := \mathcal{Q} \times \mathcal{P}$.

With these notions of momentum and force defined on $T^*\mathcal{Q}$, the Boltzmann equation for a CS probability distribution obtains as the classical reduction of the Heisenberg equation for the density operator $\boldsymbol{\rho}$. Explicitly,

$$\frac{df^{(\kappa)}(\boldsymbol{q}, \boldsymbol{\pi}; t)}{dt} := \text{tr} \frac{d\boldsymbol{\rho}^{(\kappa)}(t)}{dt} = \text{tr} \{ \boldsymbol{\rho}, H(\boldsymbol{q}, \boldsymbol{\pi}) \}^{(\kappa)} \tag{3.36}$$

where now $H = H(\boldsymbol{q}, \boldsymbol{\pi})$ is a function of the 'interaction' position and 'interaction' momentum which includes the $U(4)$ contribution to \underline{T} and presumably encodes the

³²Note that the phase space will not be a cotangent bundle in general.

classical body forces. However, more interesting than this scalar function equation is the 4×4 matrix Boltzmann equation

$$\frac{d\boldsymbol{\rho}^{(\kappa)}(t)}{dt} = \{\boldsymbol{\rho}, H(\underline{\mathbf{q}}, \underline{\boldsymbol{\pi}})\}^{(\kappa)} . \quad (3.37)$$

We do not pursue it here, but off-hand it appears to include *independent* linear and rotational classical degrees of freedom. In particular, for quasi particles of a meso/macroscopic systems, perhaps the matrix Boltzmann equation provides a handle on vortex dynamics.

4 Conclusion

There are three main pillars of SQM; dynamical $Sp(8, \mathbb{R})$ to govern evolution, a coherent state arena for observation and interpretation, and a vacuum with memory that records external interactions. Remind that we also expect $SO(9, \mathbb{R})$ to describe some physical systems, but the implications were not pursued here. We also suspect $Sp(2n, \mathbb{R})$ and $SO(2n + 1, \mathbb{R})$ for $n < 4$ may be relevant symmetries for some highly ordered meso/macroscopic systems. (There is obviously plenty of motivation to also consider $OSp(9, 8)$, and one can see some remarkable similarities to certain conjectured M-theory constructs[33] in this case.)

A rather obvious decomposition of $\mathfrak{Sp}(8, \mathbb{R})$ hints at the possibility of a nontrivial internal /external symmetry unification. According to our interpretation of the algebra decomposition, space-time intervals will be expectation values of quantum observables and Poincaré will be a limiting symmetry. In consequence, meshing Poincaré with gauge quantum mechanics is not an issue, and the no-go theorem regarding mixing internal/Lorentz symmetries does not apply. In a nutshell, our proposal is to replace relativistic quantum mechanics with symplectic quantum mechanics; eventually including $SO(9, \mathbb{R})$ and perhaps $OSp(9, 8)$ as a portal between the two.

Defining the symplectic quantum theory more or less follows standard quantum mechanics except that the group determines both observables associated with internal degrees of freedom and the kinematic observables usually associated with phase space — and the commutation relations among them. The theory is quantized by constructing the Hilbert space from induced representations and using the functional Mellin transform to transfer the algebra of observables to the operator algebra on the Hilbert space.

It is significant that the dynamics of these observables is governed by the same group they help generate. In fact, for any dynamical evolution of a system, \mathfrak{P} will in general change according to the adjoint group action. From this perspective, it is natural to guess that the ten generators contained in \mathfrak{Z}_- have something to do with inertia. Similarly, the ten generators comprising \mathfrak{Z}_+ appear to represent 10-d configuration space that we interpret as four linear dimensions and six directed-area

‘dimensions’.³³ These momentum-type and position-type operators will not remain static in general. It is tempting to interpret this as quantum gravity — as least as a toy model. Moreover, the $\mathfrak{U}(4)$ subalgebra contains 16 gauge bosons. Again it is tempting to compare the $9 + 1$ bosons coming from $U(4) \supset U(3) \times U(1)$ with the $SU(3) \times U(1)$ of the Standard model and the remaining 6 with broken $SU(2)$ massive gauge bosons and their anti-particles. There is, of course, an excess of one (supposedly massless) gauge boson.

The coherent state model facilitates physical interpretation, so it would be satisfying to develop the CS model of the dynamics as Hamiltonian mechanics on a non-commutative phase space.³⁴ Off hand, a functional integral approach seems to be indicated: but it properly deserves a detailed study and so was not included in this paper. Presumably, such a model looks similar to a QFT-like gauge theory on Z but with important differences; the most obvious being no *a priori* Poincaré symmetry and an adjustable vacuum. In consequence, the model *is not* space-time local, but it *is* Z local. Also, complicated dynamics relative to a simple vacuum and associated CS can be rendered simple dynamics by a judicious choice of a complicated vacuum and associated CS — complicated in the sense that the inducing parabolic subgroup obscures the physical interpretation in terms of elementary particles/fields. Less obvious but equally important is the fact that the coherent states can be expanded in a basis given by the discrete series representation(s): Hence, the CS model would enjoy an interpretational advantage while the discrete series would supply a proper mathematical definition.

There is a growing consensus in recent years that space-time is emergent in some sense and matrix models are perhaps more fundamental than Yang-Mills QFT (for a review see [34]). It is remarkable that the fairly simple-minded symplectic quantum mechanics leads naturally and unambiguously (modulo the initial choice of dynamical group) to similar notions. It is likely that quantum mechanics based on the sister group $SO(9, \mathbb{R})$ and parent group $OSp(9, 8)$ have more surprises in store.

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³³There are clearly implications regarding entropy content and transfer if a volume element in a 4-d space can be characterized by the six directed-area ‘dimensions’ — especially if their associated observables encode vortex-like dynamics.

³⁴It would likely be fruitful to also develop a perturbative symplectic bosonic string theory in the matrix picture employing Barut/Girardello-type CS.

References

- [1] J. LaChapelle, Functional Integral Approach to C^* -algebraic Quantum Mechanics, arXiv:math-ph/1505.08102 (2015).
- [2] J. LaChapelle, A Proposed Definition of Functional Integrals, arXiv:math-ph/1501.01602 (2015).
- [3] P. Pajas and R. Raczka, Degenerate Representations of the Symplectic Groups I. the Compact Group $Sp(n)$, *International Center for Theoretical Physics*, Trieste (1966).
- [4] D.J. Rowe and J.L. Wood, *Fundamentals of Nuclear Models: Foundational Models*, World Scientific, New Jersey, (2010).
- [5] B. Arvind, N. Dutta, and R. Simon, The Real Symplectic Groups in Quantum Mechanics and Optics, arXiv:quant-ph/9509002 (1995).
- [6] A. Wünsche, Symplectic groups in quantum optics, *J. Opt. B* **2**, (2000).
- [7] N. Krausz and M.S. Marinov, Exact evolution operator on non-compact group manifolds, *J. Math. Phys.* **41**, 5180 (2000).
- [8] Y. Choquet-Bruhat and C. DeWitt-Morette, *Analysis, Manifold and Physics*, Elsevier, Amsterdam, (1982).
- [9] J. Fuchs, and C. Schweigert, *Symmetries, Lie Algebras and Representations*, Cambridge Univ. Press, Cambridge UK, (2003).
- [10] Jin-Quan Chen, Jialun Ping, and Fan Wang, *Group Representation Theory for Physicists*, World Scientific, New Jersey, (2002).
- [11] S. Coleman and J. Mandula, All Possible Symmetries of the S Matrix, *Phys. Rev.* **159**(5), 1251 (1967).
- [12] J. Deenen and C. Quesne, Partially coherent states of the real symplectic group, *J. Math. Phys.* **25**(8), 2354 (1984).
- [13] S.D. Bartlett, D.J. Rowe, and J. Repka, Vector coherent state representations, induced representations, and geometric quantization: II. Vector coherent state representations, arXiv:quant-ph/0201130v2.
- [14] D.P. Williams, *Crossed Products of C^* -algebras*. American Mathematical Society, Providence, Rhode Island, (2007).
- [15] J. LaChapelle, Exploring Functional Mellin Transforms, arXiv:math-ph/1501.01889 (2015).

- [16] N.P. Landsman, Rieffel induction as generalized quantum Marsden-Weinstein reduction, *J. Geo. and Phys.* **15**, 285-319, (1995).
- [17] G.W. Mackey, On induced representations of groups, *Amer. J. Math.* **73**, 576–592, (1951).
- [18] G.W. Mackey, Induced representations of locally compact groups. I., *Ann. of Math.* **55**(2) 101–139, (1952).
- [19] G.W. Mackey, Induced representations of locally compact groups. II. The Frobenius reciprocity theorem, *Ann. of Math.* **58**(2) 193–221, (1953).
- [20] R.C. King and B.G. Wybourne, Holomorphic discrete series and harmonic series unitary irreducible representations of non-compact Lie groups: $Sp(2n, \mathbb{R})$, $U(p, q)$, and $SO^*(2n)$, *J. Phys. A: Math. Gen.* **18**, 3113 (1985).
- [21] I.M. Gelfand and M.I. Graev, *Am. Math. Soc. Transl. Ser. 2* **164**, 116–46 (1967).
- [22] Harish-Chandra, Discrete series for semisimple Lie groups: I, *Acta Math.* **113**, 241–318 (1965), and Discrete series for semisimple Lie groups: II, *Acta Math.* **116**, 1-111 (1966).
- [23] H. Hecht and W. Schmid, A proof of Blattner’s conjecture, *Inventiones Math.* **31**, 129–154 (1975).
- [24] M. Atiyah and W. Schmid, A geometric construction for the discrete series for semisimple Lie groups, *Inventiones Math.* **42**, 1–62 (1977).
- [25] P. Kramer and Z. Papadopolos, Hilbert spaces of analytic functions and representations of the positive discrete series of $Sp(6, \mathbb{R})$, *J. Phys. A: Math. Gen.* **19**, 1083–1092 (1986).
- [26] R. Camporesi, Harmonic Analysis and Propagators on Homogeneous Spaces, *Phys. Rep.* **196**, 1–134 (1990).
- [27] V.I. Arnol’d and A.B. Givental’, Symplectic Geometry, in *Dynamical Systems IV*, Springer-Verlag, New York (1990).
- [28] J. Wei and E. Norman, On global representations of the solutions of linear differential equations as a product of exponentials, *Proc. Am. Math. Soc.* **4**(4), 575–581 (1963).
- [29] R. Gilmore, *Lie Groups, Lie Algebras, and Some of Their Applications*. Dover Publications, New York (2002).
- [30] J. LaChapelle, Functional Integration on Constrained Function Spaces I and II, arXiv:math-ph/1212.0502 (2012) and arXiv:math-ph/1405.0461 (2014).

- [31] P. Cartier and C. DeWitt-Morette, *Functional Integration: Action and Symmetries*. Cambridge University Press, Cambridge (2006).
- [32] M. Blau and G. Thompson, Localization and Diagonalization, *J. Math. Phys.* **36**, 2192 (1995).
- [33] T. Banks, W. Fischler, S.H. Shenker, and L. Susskind, M Theory as a Matrix Model: a Conjecture, *Phys. Rev. D* **55**, 5112 (1997).
- [34] H. Steinacker, Emergent geometry and gravity from matrix models: an introduction, HAL Id: hal-00606646 (2011).